CARLEMAN ESTIMATES FOR THE KORTEWEG-DE VRIES EQUATION WITH PIECEWISE CONSTANT MAIN COEFFICIENT

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Abstract. In this article, we investigate observability-related properties of the Korteweg-de Vries equation with a discontinuous main coefficient, coupled by suitable interface conditions. The main result is a novel two-parameter Carleman estimate for the linear equation with internal observation, assuming a monotonicity condition on the main coefficient. As a primary application, we establish the local exact controllability to the trajectories by employing a duality argument for the linear case and a local inversion theorem for the nonlinear equation. Secondly, we establish the Lipschitz-stability of the inverse problem of retrieving an unknown potential using the Bukhgeĭm-Klibanov method, when some further assumptions on the interface are made. We conclude with some remarks on the boundary observability.

1. Introduction

Let $T>0$, $L>0$ and $p:[0,L]\to\mathbb{R}^+$ be a piecewise constant function where $p(x)=p_k>0$ on each $[a_k,a_{k+1})$ with $\Gamma=\{a_1<a_2<\cdots<a_{N-1}\}$ and $a_0=0$, $a_N=L$. Given some initial data y_0, we study the following Korteweg-de Vries equation with piecewise constant main coefficient

$$\begin{cases} y_t+p(x)y_{xxx}+y_x+yy_x=0, & (t,x)\in(0,T)\times(0,L),\\ y(t,0)=y(t,L)=y_x(t,L)=0, & t\in(0,T),\\ y(0,x)=y_0(x), & x\in(0,L),\end{cases}\tag{1.1}$$

coupled by the *transmission conditions*

$$\begin{cases} y(t,a_k^-) = y(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!],\\ \sqrt{p_{k-1}}y_x(t,a_k^-) = \sqrt{p_k}y_x(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!],\\ p_{k-1}y_{xx}(t,a_k^-) = p_ky_{xx}(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!].\end{cases}\tag{TC}$$

Despite the discontinuity of the coefficient p, the transmission conditions allow us to consider this model as a whole in $(0,T)\times(0,L)$, since they act as boundary conditions on each (a_k,a_{k+1}), $k\in[\![0,N-1]\!]$.

The Korteweg-de Vries equation (KdV) is a well-known dispersive equation, introduced by Korteweg-de Vries [KDV95] to model the propagation of water waves in a shallow channel, namely, water waves with small amplitude and large wavelength compared to the undisturbed depth profile. In this context, this kind of water waves in a channel with a sudden jump in the depth profile have been modeled by a KdV equation with a discontinuous main coefficient [PV92]. Thus, in this model the function $p=p(x)$ can be interpreted as the undisturbed depth profile with jumps of the channel, whereas $p(x)+y(t,x)$ represents the wave surface at time t at position x. On the mathematical side, to solve the KdV equation on the half-line, Deconinck, Sheils and Smith [DSS16] has proposed (several) interface conditions involving the first three derivatives (in space) of the solution at the interface. The specific interface conditions of system (1.1)-(TC) were proposed by Crépeau [Cré16], where the exact boundary controllability of such a model is studied. In this work, we continue the

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study of this equation by obtaining a new Carleman estimate under a monotonicity hypothesis on the main coefficient, allowing us to deduce some results on the controllability and the recovery of some parameter for this equation.

1.1. **Main results.** Let us define $I_k := (a_k, a_{k+1})$, $k \in [\![0, N-1]\!]$. In what follows ω will be always a non-empty open subset of $(0, L)$. Due to the discontinuity of the main coefficient, it is natural to impose some restriction on where the observation zone should be located. In this direction we will introduce the following assumption on the pair (ω, p):

Hypothesis $\mathfrak{M}$. there exists $j \in \{0, \dots, N-1\}$ such that the observation zone ω is such that $\overline{\omega} \subset I_j$, henceforth denoted as $\omega \Subset I_j$, and the following monotonicity property holds depending on the value of j:

(1) if $j \notin \{0, N-1\}$ then

$$\begin{cases} p(a_k^-) > p(a_k^+), \ \ k \in [\![1, j]\!], \\ p(a_k^-) < p(a_k^+), \ \ k \in [\![j+1, N-1]\!]; \end{cases}$$

(2) if $j = 0$ then

$$p(a_k^-) < p(a_k^+), \ \ k \in [\![1, N-1]\!];$$

(3) if $j = N-1$ then

$$p(a_k^-) > p(a_k^+), \ \ k \in [\![1, N-1]\!].$$

Let $s \geq 0$ and let us introduce the space

$$H^s_\Gamma(0, L) := \{u \in L^2(0, L) \mid u_{|_{I_k}} \in H^s(I_k), \ \ k \in [\![0, N-1]\!]\}. \tag{1.2}$$

Via isomorphism, $H^s_\Gamma(0, L)$ can be seen as the direct sum of the Sobolev spaces $H^s(I_k)$ for $k \in [\![0, N-1]\!]$. Thus, it has a Hilbert space structure equipped the inner product

$$\langle u, v\rangle_{H^s_\Gamma(0,L)} = \sum_{k=1}^{N-1} \langle u_{|_{I_k}}, v_{|_{I_k}}\rangle_{H^s(I_k)}.$$

1.1.1. *Carleman estimate.* Let $Q := (0, T) \times (0, L)$. For some ρ_0, $\rho_1 > 0$ given, let us assume that

$$\rho_0 \leq \min_{x\in[0,L]} p(x) \ \ \text{and} \ \ \max_{x\in[0,L]} p(x) \leq \rho_1.$$

Let b, $d \in L^\infty(Q)$ and let $\mathcal{L} : \mathcal{V} \to L^2(Q)$ be the differential operator given by

$$\mathcal{L} = \partial_t + p(x)\partial_x^3 + b(t, x)\partial_x + d(t, x) \tag{1.3}$$

where $\mathcal{V}$ is the space of functions $u \in L^2(0, T; H^3_\Gamma(0, L) \cap H^1_0(0, L))$ such that $\mathcal{L}u \in L^2(Q)$ and u satisfies the transmission conditions

$$\left\{ \begin{array}{rcll} u(t, a_k^-) &=& u(t, a_k^+), & t \in (0, T), \ \ k \in [\![1, N-1]\!], \\ \sqrt{p_{k-1}}u_x(t, a_k^-) &=& \sqrt{p_k}u_x(t, a_k^+), & t \in (0, T), \ \ k \in [\![1, N-1]\!], \\ p_{k-1}u_{xx}(t, a_k^-) &=& p_k u_{xx}(t, a_k^+), & t \in (0, T), \ \ k \in [\![1, N-1]\!]. \end{array} \right. \tag{1.4}$$

For ω_0 be a non-empty open subset of ω such that $\omega_0 \Subset \omega$. Fix $\kappa \in (1, 2)$ and let β be the weight function constructed by Lemma 3.1 below. For $\lambda > 0$, we introduce the Carleman weights

$$\eta(t, x) = \frac{e^{\kappa\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)}}{t(T-t)} \ \ \text{and} \ \ \xi(t, x) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \tag{1.5}$$

for $(t, x) \in Q$. The main result is the following two-parameter Carleman estimate.

Theorem 1.1. *Let (ω, p) satisfy Hypothesis $\mathfrak{M}$ and let $\omega_0 \Subset \omega$ be non-empty and open. There exist $s_0 > 0$, $\lambda_0 > 0$ and a constant $C > 0$ depending on ω, Γ, L, T, p, $\|\beta\|_{C^3([0,L]\setminus\Gamma)}$, s_0 and λ_0 such that for all $u \in \mathcal{V}$ we have*

$$C \iint_Q e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2 + s^3\lambda^4\xi^3|u_x|^2 + s\lambda^2\xi|u_{xx}|^2)dxdt$$
$$\leq \|e^{-s\eta}\mathcal{L}u\|^2_{L^2(Q)} + \iint_{(0,T)\times\omega} e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2dxdt + s^3\lambda^4\xi^3|u_x|^2dxdt + s\lambda^2\xi|u_{xx}|^2)dxdt \quad (1.6)$$

for any $s \geq s_0$ and $\lambda \geq \lambda_0$.

The above estimate is derived by following Fursikov and Imanuvilov [FI96]. We emphasize that a Carleman estimate is of interest in itself due to the many applications they have found. We now give two applications and in Section 6 we briefly discuss the case of boundary observability.

1.1.2. *Local controllability to the trajectories.* Let us consider the controlled KdV equation

$$\begin{cases} y_t + p(x)y_{xxx} + y_x + yy_x = \mathbb{1}_\omega v, & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \quad (1.7)$$

coupled by the transmission conditions (TC), where y_0 is the initial condition, and v is a control localized in some non-empty open set $\omega \subset (0,L)$. We are interested in the *exact controllability to the trajectories* for the KdV equation (1.7). More precisely, we wonder if, given $T > 0$ and a solution $\overline{y}$ of the uncontrolled KdV equation

$$\begin{cases} \overline{y}_t + p(x)\overline{y}_{xxx} + \overline{y}_x + \overline{y}\overline{y}_x = 0, & (t,x) \in (0,T)\times(0,L), \\ \overline{y}(t,0) = \overline{y}(t,L) = \overline{y}_x(t,L) = 0, & t \in (0,T), \\ \overline{y}(0,x) = \overline{y}_0(x), & x \in (0,L), \end{cases} \quad (1.8)$$

coupled by the corresponding transmission conditions (TC), there exists a control $v = v(t)$ such that the corresponding solution $y = y(t,x)$ satisfies $y(T,\cdot) = \overline{y}(T,\cdot)$ on $(0,L)$. In Section 2 we discuss the well-posedness of systems (1.7) and (1.8), both coupled by (TC). The controllability result is the following.

Theorem 1.2. *Let $T > 0$ and let (ω, p) satisfy Hypothesis $\mathfrak{M}$. If $\overline{y} \in C([0,T], L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ is the solution of (1.8), then there exists $\delta > 0$ such that for any $y_0 \in L^2(0,L)$ satisfying $\|y_0 - \overline{y}_0\|_{L^2(0,L)} \leq \delta$, we can find a control $v \in L^2((0,T)\times\omega)$ such that the corresponding solution y to (1.7) satisfies*

$$y(T,\cdot) = \overline{y}(T,\cdot) \quad \text{in} \quad (0,L).$$

The strategy consists in first consider the system satisfied by $z = y - \overline{y}$, which is given by

$$\begin{cases} z_t + p(x)z_{xxx} + (\overline{y}z)_x + zz_x = \mathbb{1}_\omega v, & (t,x) \in (0,T)\times(0,L), \\ z(t,0) = z(t) = z_x(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \quad (1.9)$$

coupled by the corresponding transmission conditions (TC). Then, establishing the exact control to the trajectories reduces to establish the null controllability of the system (1.9). This is done by studying the null controllability of the linearization of (1.9) and then employing a fixed point argument to treat the nonlinear system. The null controllability of the linearized system follows by a duality argument and a suitable observability estimate for the adjoint system. This observability estimate is derived from the Carleman estimate given in Theorem 1.1.

1.1.3. *Retrieving a potential term.* We consider the following nonlinear KdV equation

$$\begin{cases} y_t + p(x)y_{xxx} + \mu y_x + yy_x = 0, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1.10}$$

coupled by the transmission conditions (TC), with potential $\mu = \mu(x)$ and initial data y_0. We denote its solution by $y = y[\mu]$. We make the following assumptions on the interface Γ.

Hypothesis $\mathfrak{I}$. The interface Γ and the coefficient p satisfy:

- The middle point of the domain is not an interface point: $a_k \neq L/2$ for each $k \in [\![0, N-1]\!]$.
- The interface is symmetric, in the sense that $a_k + a_{N-k} = L$ for all $k \in [\![0, N]\!]$.

For $m > 0$ given, let us introduce the set of admissible potentials

$$\mathfrak{P}^{sym}_{\leq m}(0,L) = \{\mu \in L^\infty(0,L) \mid \mu(x) = \mu(L-x), \forall x \in [0,L] \text{ and } \|\mu\|_{L^\infty} \leq m\}.$$

Before presenting the result, we refer to Section 2 below for the definition of the space $\mathcal{H}^3_\Gamma(0,L)$. Following the Bukhgeĭm-Klibanov method, we can employ a slight variant of the Carleman estimate given in Theorem 1.1 to establish the Lipschitz continuity of the inverse problem consisting on retrieving the potential term on the equation (1.10). The result is the following.

Theorem 1.3. *Let $\omega \subset (0,L)$ be a nonempty open set containing $L/2$. Assume that Γ satisfy Hypothesis $\mathfrak{I}$. Let p be symmetric with respect to $L/2$, that is, $p(x) = p(L-x)$ for all $x \in [0,L]$ and*

$$\begin{cases} p(a_k^-) > p(a_k^+), & k \in [\![1, \lfloor N/2 \rfloor]\!], \\ p(a_k^-) < p(a_k^+), & k \in [\![\lfloor N/2 \rfloor, N]\!]. \end{cases}$$

Let $\omega \subset (0,L)$ be a nonempty open set containing $L/2$. Assume that Γ satisfy Hypothesis $\mathfrak{I}$.

Let m, r_0 and K be some given positive constants. Let $y_0 \in H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L)$ satisfying

$$y_0'(x) = y_0'(L-x) \text{ and } |y_0'(x)| \geq r_0 > 0, \ \forall x \in [0,L].$$

There exists a positive constant C depending on L, T, Γ, ω, m, r_0 and K such that for all μ, $\nu \in \mathfrak{P}^{sym}_{\leq m}(0,L)$, it holds

$$\|\mu - \nu\|_{L^2(0,L)} \leq C\|y - z\|_{H^1(0,T;H^2_\Gamma(\omega))},$$

where the corresponding solutions $y = y[\mu]$ and $z = z[\nu]$ of (1.10)-(TC) issued from y_0 satisfy

$$\max\{\|y\|_{W^{1,\infty}(0,T;W^{1,\infty}(0,L))}, \|z\|_{W^{1,\infty}(0,T;W^{1,\infty}(0,L))}\} \leq K.$$

Remark 1.4. Along the proof, we apply the Carleman estimate for (ω_0, p) satisfying Hypothesis $\mathfrak{M}$, where $\omega_0 \Subset \omega$ is symmetric with respect to $L/2$. The symmetry and monotonicity hypothesis imposed on p is then compatible with Hypothesis $\mathfrak{I}$.

1.2. **Some comments on the literature.** The Korteweg-de Vries equation is one of the most celebrated nonlinear dispersive equations. The study of its controllability properties began with the early works of Russel and Zhang [RZ93, RZ96]. Since then, extensive research has been conducted on its controllability properties. A good survey of results up to 2014 is provided by Cerpa [Cer14]. Here, we briefly highlight some key issues.

Regarding our setting, where we aim to study the control properties of the equation, the uncontrolled system (1.1)-(TC), which notably has a piecewise constant dispersion coefficient, was first proposed by Crépeau [Cré16]. In that article, the boundary exact controllability with a single control acting on the Neumann boundary condition is established by a multiplier technique, under certain (smallness) conditions involving the coefficient p, the time $T > 0$ and the length $L > 0$. To the best of the author's knowledge, this is the only controllability result for such an equation. The exact boundary controllability of the KdV equation is a delicate issue, as was already noticed by

Rosier [Ros97] when he established (by a perturbative approach) that the exact controllability with right Neumann control holds if the length L does not belong to a certain set of critical lengths. Later, using more refined nonlinear methods, Coron and Crépeau [CC04] showed the (local) exact controllability of the nonlinear system holds even when the length is critical. Since then, extensive research has been conducted on various control problems surrounding this issue; a good survey of this phenomenon is given by Capistrano-Filho [CF24]. As pointed out by Crépeau, obtaining exact control properties in the discontinuous setting, even by a perturbative approach, appears to be a challenging problem.

A less demanding property is the control to the trajectories. This was studied by Glass and Guerrero [GG08] in the case of boundary controls. Observe that when $\overline{y} = 0$, the control to the trajectories is known as null controllability. Thus, the control to the trajectories can be seen as a result in between the null controllability and exact controllability of the system. Moreover, it has been used as a stepping stone to obtain exact control properties by introducing additional controls acting on the system. For instance, Chapouly [Cha09] employed this approach, leveraging control properties of the viscous Burgers equation.

Furthermore, the control properties of linear KdV equation in different settings have already been addressed by some authors. On one hand, the uniform controllability in the vanishing dispersion limit has been addressed by Glass and Guerrero [GG08, GG09] under different boundary conditions. On the other hand, the controllability properties of the KdV equation in networks with various configurations has caught significant attention in recent years; see, for example, Capistrano-Filho, Parada and da Silva [FPdS25] and the references therein.

With regard to global Carleman estimates for the KdV equation, when a variable main coefficient is considered Baudouin, Cerpa, Crépeau and Mercado [BCCM14] derived a Carleman estimate for a sufficiently regular main coefficient, leading to Lipschitz stability in the inverse problem of retrieving the main coefficient of the equation. Here, we extend this result in a certain sense by allowing discontinuities in the main coefficient.

Aiming to obtain controllability results and Lipschitz stability for certain inverse problems, global Carleman estimates for PDEs with discontinuous principal coefficients have been derived in various contexts. Most of these, however, impose either a monotonicity condition on the jump of the principal coefficient, strong geometric assumptions at the interface, or both. Given the extensive literature, we mention the early works addressing the heat equation by Doubova, Osses, and Puel [DOP02], the wave equation by Baudouin, Mercado, and Osses [BMO07], and the Schrödinger equation by Baudouin and Mercado [BM08]. We also note that the monotonicity condition on the principal coefficient has been relaxed in some one-dimensional cases: see Benabdallah, Dermenjian, and Le Rousseau [BDLR07] for the heat equation, and Imba [Imb25] for the wave equation.

We emphasize that local Carleman estimates are a key tool in establishing unique continuation, observability, and controllability results. Moreover, their derivation typically requires less restrictive geometric conditions, in sharp contrast to the global case. For example, without any monotonicity assumptions or geometric conditions on the interface, local Carleman estimates (and related applications) were studied by Léautaud, Le Rousseau, and Robbiano [LRLR13] in the multi-dimensional parabolic case, and by Filippas [Fil24] for the multi-dimensional wave equation.

1.3. **Outline of the paper.** The rest of the article is organized as follows. In Section 2 we establish the well-posedness of the uncontrolled KdV equation (1.7)-(TC) along with some regularity results for the linear KdV as well as for its adjoint. In Section 3 we derive a new two parameter Carleman estimate. In Section 4 we prove the control to the trajectories of the KdV equation. In Section 5 we prove the Lipschitz stability of the inverse problem of retrieving a potential term. In Section 6 we give final remarks about boundary observability.

1.3.1. *Notation.* To make our computations clearer, the symbol $[\cdot]_a$ will denote the jump at $a \in \Gamma$, namely, for a function μ we write $[\mu]_a := \mu(a^+) - \mu(a^-)$. Given two quantities X and Y, we will employ the notation $X \lesssim Y$ to say that $X \le CY$ for some $C > 0$, possibly depending on several parameters involved in the computations. Often, we will use such notation when the constant does not matter on the analysis or when its dependency is understood.

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2. Well-posedness results

Let us first recall the definition (1.2) of $H^3_\Gamma(0,L)$ for $s = 3$. By Gagliardo-Nirenberg's inequality, if $u_{|_{I_k}} \in H^3(I_k)$, it also belongs to $C^2(\overline{I_k})$ and the operator

$$u_{|_{I_k}} \in H^3(I_k) \longmapsto u_{|_{I_k}} \in C^2(\overline{I_k}) \tag{2.1}$$

is continuous. In particular, any $u \in H^3_\Gamma(0,L)$ belongs to $C^2(\overline{I_k})$ for each $k \in [\![1, N-1]\!]$. For $u \in H^3_\Gamma(0,L)$, we introduce the transmission conditions

$$\begin{cases} u(a_k^-) &= u(a_k^+), & k \in [\![1, N-1]\!], \\ \sqrt{p_{k-1}}u'(a_k^-) &= \sqrt{p_k}u'(a_k^+), & k \in [\![1, N-1]\!], \\ p_{k-1}u''(a_k^-) &= p_k u''(a_k^+), & k \in [\![1, N-1]\!]. \end{cases} \tag{2.2}$$

The natural space for the study of the system (1.1)-(TC) is defined as

$$\mathcal{H}^3_\Gamma(0,L) = \{u \in H^3_\Gamma(0,L) \mid u(0) = u(L) = u'(L) = 0 \ \text{ and } u \text{ satisfies (2.2)}\},$$

which is a closed subspace of H^3_Γ and therefore a Hilbert space endowed with the inherited inner product of H^3_Γ.

Additionally, let us introduce the Banach space

$$\mathcal{X}^0_T(0,L) = C([0,T], L^2(0,L)) \cap L^2(0,T; H^1(0,L)),$$

equipped with the norm

$$\|\cdot\|_{\mathcal{X}^0_T(0,L)} = \|\cdot\|_{L^2(0,T;L^2(0,L))} + \|\cdot\|_{L^2(0,T;H^1(0,L))}.$$

Similarly, we introduce the space

$$\mathcal{X}^3_{\Gamma,T}(0,L) = C([0,T], \mathcal{H}^3_\Gamma(0,L)) \cap L^2(0,T; \mathcal{H}^3_\Gamma \cap H^4_\Gamma(0,L)),$$

equipped with the natural norm.

2.1. **Linear Cauchy problem.** Let us consider the inhomogeneous Korteweg-de Vries equation with source term f

$$\begin{cases} y_t + p(x)y_{xxx} = f, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{2.3}$$

coupled along with the transmission conditions (TC).

Let $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L)$ be the formally defined linear operator given by $\mathcal{A} = -p(x)\partial_x^3$ with domain $\mathrm{dom}(\mathcal{A}) := \mathcal{H}^3_\Gamma(0,L)$. We also introduce its formal adjoint operator of $\mathcal{A}$, defined by $\mathcal{A}^* : z \mapsto p(x) z_{xxx}$ with domain $\text{dom}(\mathcal{A}^*)$ given by those functions $z \in H^3_\Gamma(0,L)$ satisfying $z(0) = z(L) = z_x(0) = 0$ and the transmission conditions (2.2).

Lemma 2.1. *The operators $\mathcal{A}$ and $\mathcal{A}^*$ are well-defined.*

Proof. Let $z \in \mathcal{H}^3_\Gamma(0,L)$. If $\varphi \in C^\infty_c(0,L)$, as $z \in H^3(I_k)$ for all $k \in [\![0, N-1]\!]$, by performing integration by parts on each I_k and adding up all these integrals, we get

$$\sum_{k=0}^{N-1} \int_{a_k}^{a_{k+1}} p\partial_x^3 z \varphi dx = \sum_{k=1}^{N} \left(\int_{a_{k-1}}^{a_k} p_{k-1} z_{xx} z_x dx - p_{k-1} z_{xx} z \Big|_{a_{k-1}}^{a_k} \right)$$
$$= -\int_0^L (p\partial_x^2 z)(\partial_x \varphi) dx + \sum_{k=1}^{N-1} \varphi(a_k) [p_k z_{xx}]_{a_k}.$$

First, note that $[p_k z_{xx}]_{a_k} = 0$ for $k \in [\![1, N-1]\!]$ due to the transmission conditions and therefore the trace terms above all vanish. As $p\partial_x^2 z \in L^2(0,L)$, we have that $\langle p\partial_x^3 z, \varphi \rangle_{L^2(0,L)}$ is well-defined and equals $-\int_0^L (p\partial_x^2 z)(\partial_x \varphi) dx$. As φ is arbitrary, $p\partial_x^3 z$ is well-defined as a function in $L^2(0,L)$ and the conclusion follows. □

We can now employ semigroup theory tools to study the linear Cauchy problem (2.3)-(TC).

Proposition 2.2. *The operators $\mathcal{A}$ and $\mathcal{A}^*$ both generate a strongly continuous semigroup of contractions on $L^2(0,L)$.*

Proof. The operators $\mathcal{A}$ and $\mathcal{A}^*$ are both closed. If $z \in \mathcal{D}(\mathcal{A})$ then

$$\begin{aligned}
\langle \mathcal{A}z, z \rangle_{L^2(0,L)} &= -\int_0^L p(x) z_{xxx} z dx \\
&= \sum_{k=1}^{N} \left(\int_{a_{k-1}}^{a_k} p_{k-1} z_{xx} z_x dx - p_{k-1} z_{xx} z \Big|_{a_{k-1}}^{a_k} \right) \\
&= \sum_{k=1}^{N} \left(\frac{p_{k-1}}{2} \left(|z_x(a_k^-)|^2 - |z_x(a_{k-1}^-)|^2 \right) - p_{k-1} z_{xx}(a_k^-) z(a_k^-) + p_{k-1} z_{xx}(a_{k-1}^+) z(a_{k-1}^+) \right) \\
&= -\frac{p_0}{2} |z_x(0)|^2 + \sum_{k=1}^{N-1} \left(-\frac{1}{2} [p|z_x|^2]_{a_k} + z(a_k)[p_k z_{xx}]_{a_k} \right) \\
&= -\frac{p_0}{2} |z_x(0)|^2 \leq 0.
\end{aligned}$$

In a similar way, $\mathcal{A}^*$ is also dissipative since

$$\langle z, \mathcal{A}^* z \rangle_{L^2(0,L)} = -\frac{p_{N-1}}{2} |z_x(L)|^2 \leq 0.$$

The conclusion follows from the classical Lumer-Phillips Theorem. □

Using semigroup theory and the multipliers method, we obtain the following well-posedness result for the linear Cauchy problem (2.3)-(TC) along with the classical Kato smoothing effect.

Proposition 2.3. *Let $T > 0$. If $y_0 \in L^2(0,L)$ and $f \in L^1(0,T; L^2(0,L))$, then there exists a unique mild solution y of the KdV equation (2.3)-(TC) that belongs to $\mathcal{X}^0_T(0,L)$. Also, there exists $C = C(T, L, \Gamma, p) > 0$ such that*

$$\|y\|_{\mathcal{X}^0_T(0,L)} \leq C\big(\|y_0\|_{L^2(0,T)} + \|f\|_{L^1(0,T;L^2(0,L))} \big).$$

Proof. Using Proposition 2.2, there exists a unique mild solution y of (2.3) which belongs to $C([0,T],L^2(0,L))$, see [Paz83, Chapter 4]. Moreover, there exists $C>0$ such that

$$\|y\|^2_{C([0,T],L^2(0,L))} \leq C(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1(0,T;L^2(0,L))}).$$

In what follows we consider $y_0 \in \mathcal{H}^3_\Gamma(0,L)$ which, by a standard density argument, is enough to employ the multiplier method. Let $q \in C([0,T]\times[0,L])$ be such that $q_k \in C^\infty([0,T]\times\overline{I_k})$ where q_k denotes the restriction of q to I_k for $k \in [\![0,N-1]\!]$. Performing several integration by parts we obtain

$$2\int_0^s\int_{I_k} qyy_t dxdt = -\int_0^s\int_{I_k} q_t|y|^2dxdt + \int_{I_k} q|y|^2\Big|_0^s dx$$

and

$$2\int_0^s\int_{I_k} qpyy_{xxx}dxdt = -\int_0^s\int_{I_k} pq_{xxx}|y|^2dxdt + 3\int_0^s\int_{I_k} pq_x|y_x|^2dxdt$$
$$+\int_0^s (pq_{xxx}|y|^2 - pq|y_x|^2 - 2pq_xyy_x + 2pqyy_{xx})\Big|_{a_k}^{a_{k+1}} dt,$$

for each $k \in [\![0,N-1]\!]$. By adding these equations we get

$$3\int_0^s\int_0^L pq_x|y_x|^2dxdt + \int_0^L q|y|^2\Big|_0^s dx = \int_0^s\int_0^L (q_t + pq_{xxx})|y|^2dxdt$$
$$+2\int_0^s\int_0^L qyf dxdt - \int_0^s p_0|y_x(t,0)|^2dt + \sum_{k=1}^{N-1}\int_0^s([q+pq_{xx}]_{a_k}|y(t,a)|^2 - [q]_{a_k}p_k|y_x(t,a_k^+)|^2)dt$$
$$+2\sum_{k=1}^{N-1}\int_0^s([q]_{a_k}y(t,a_k)p_ky_{xx}(a_k^+) - [\sqrt{p}q_x]_{a_k}y(a_k)\sqrt{p_k}y_x(t,a_k^+))dt.$$

By setting $s=T$ and choosing $q_0(x) = x/\sqrt{p_0}$ for $x \in I_0$ and $q_k(x) = (x-a_k)/\sqrt{p_k} + q_{k-1}(a_k^-)$ for $x \in I_k$ for all $k \in [\![1,N-1]\!]$, we readily get the identity

$$3\int_0^T\int_0^L \sqrt{p}|y_x|^2dxdt + \int_0^L q|y(T,x)|^2dx = \int_0^L q|y_0(x)|^2dx + 2\int_0^T\int_0^L qyf dxdt.$$

The above identity together with the previous estimates implies

$$\|y_x\|^2_{L^2(L^2)} \leq C(T,L,\Gamma,p)\big(\|y_0\|^2_{L^2} + \|f\|^2_{L^1(L^2)}\big).$$

This concludes the proof. ☐

Remark 2.4. The multiplier employed in the previous proof was introduced by Crépeau [Cré16], capturing the transmission conditions and being similar in spirit to the one used by Rosier [Ros97].

Remark 2.5. At the L^2-level of regularity, the transmission conditions (TC) does not make sense. They are understood as an extension of the related semigroup and the solution is understood in the sense of Duhamel's formula.

2.2. **Nonlinear system.** Let $z \in \mathcal{X}^0_T(0,L)$ be given and consider the system

$$\begin{cases} y_t + p(x)y_{xxx} + y_x = -zz_x, & (t,x)\in(0,T)\times(0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t\in(0,T), \\ y(0,x) = y_0(x) & x\in(0,L), \end{cases} \tag{2.4}$$

coupled by the transmission conditions (TC). Let $\widetilde{\mathcal{A}} := -p(x)\partial_x^3 - \partial_x$ with $\mathcal{D}(\widetilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$. Straightforward computations reveal that the conclusions of Proposition 2.3 still hold true when $\mathcal{A}$ is replaced by $\widetilde{\mathcal{A}}$. Thus, due to the Kato-smoothing effect and Proposition A2, $-zz_x$ is allowed as a source term and we can introduce $\mathcal{F} : \mathcal{X}_T^0(0,L) \to \mathcal{X}_T^0(0,L)$ to be the map defined by $\mathcal{F}(z) = y$, where y is the solution of (2.4). It is well-defined and characterized by Duhamel's formula

$$y(t) = e^{t\widetilde{\mathcal{A}}}y_0 + \int_0^t e^{(t-s)\widetilde{\mathcal{A}}}(-zz_x)(s)ds, \ \forall t \in [0,T], \tag{2.5}$$

where $\left(e^{t\widetilde{\mathcal{A}}}\right)_{t\in[0,T]}$ is the strongly continuous semigroup of contractions generated by $\widetilde{\mathcal{A}}$.

We now establish global well-posedness using a classical fixed-point argument. Assuming more regularity on the initial data, we can guarantee the existence of a more regular solution as classically done for the KdV equation by Bona-Sun-Zhang [BSZ03, Theorem 4.1].

Proposition 2.6. *Let $T > 0$ and $L > 0$. For any initial condition $y_0 \in L^2(0,L)$ the system (1.1)-(TC) has a unique solution $y \in \mathcal{X}_T^0(0,L)$ which satisfies*

$$\|y\|_{\mathcal{X}_T^0(0,L)} \leq C\|y_0\|_{L^2(0,L)}, \tag{2.6}$$

for some $C = C(T, L, p, \|y_0\|_{L^2(0,L)}) > 0$. If, additionally, $y_0 \in \mathcal{H}_\Gamma^3(0,L)$ then $y \in \mathcal{X}_{\Gamma,T}^3(0,L)$ and

$$\|y\|_{\mathcal{X}_{\Gamma,T}^3(0,L)} \leq C\|y_0\|_{H_\Gamma^3(0,L)}, \tag{2.7}$$

Proof. Observe that the constant $C > 0$ given by Proposition 2.3 is affine on $T > 0$. Then, any $0 < \tau \leq T$, we have

$$\|\mathcal{F}(z)\|_{\mathcal{X}_\tau^0(0,L)} \leq C\left(\|y_0\|_{L^2(0,L)} + \|zz_x\|_{L^1(0,\tau;L^2(0,L))}\right).$$

By using Lemma A1 on $(0,L)$, the previous estimate implies

$$\|\mathcal{F}(z)\|_{\mathcal{X}_\tau^0(0,L)} \leq C\|y_0\|_{L^2} + C_1(\tau^{1/2} + \tau^{1/3})\|z\|_{\mathcal{X}_\tau^0(0,L)}\|z\|_{\mathcal{X}_\tau^0(0,L)}.$$

for some $C_1 > 0$. Let $R > 0$ be such that

$$R = 2C\|y_0\|_{L^2(0,L)} \ \text{ and } \ C_1(\tau^{1/2} + \tau^{1/3})R \leq \frac{1}{4}.$$

With this choice, for small τ, we readily see that $\mathcal{F}$ maps the closed ball $\mathbb{B}_R = \{z \in \mathcal{X}_\tau^0 : \|z\|_{\mathcal{X}_\tau^0(0,L)} \leq R\}$ into itself and

$$\begin{aligned}\|\mathcal{F}(z_1) - \mathcal{F}(z_2)\|_{\mathcal{X}_\tau^0(0,L)} &\leq C_1(\tau^{1/2} + \tau^{1/3})(\|z_1\|_{\mathcal{X}_\tau^0(0,L)} + \|z_2\|_{\mathcal{X}_\tau^0(0,L)})\|z_1 - z_2\|_{\mathcal{X}_\tau^0(0,L)} \\ &\leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{X}_\tau^0(0,L)}.\end{aligned}$$

The smallness condition imposed previously by R allows us to apply the Banach fixed-point theorem in time $\tau \leq T$, which give us a unique fixed point y of $\mathcal{F}$ belonging to $\mathbb{B}_R$ and by consequence being the unique solution $y \in \mathcal{X}_\tau^0(0,L)$ of (1.8). We observe that $\tau \in (0,T]$ is independent on $\|y_0\|_{L^2(0,L)}$. Thus, up to shrinking τ if necessary so that $n\tau = T$ for some $n \in \mathbb{N}$, we can extend the previous argument on intervals $(\tau, 2\tau], (2\tau, 3\tau], \ldots, ((n-1)\tau, n\tau = T]$. Therefore, the existence of a solution $y \in \mathcal{X}_T^0(0,L)$ is guaranteed.

If $y_0 \in \mathcal{H}_\Gamma^3(0,L)$, we can set $z := y_t$ and look at the equation it satisfies. We can replicate the previous reasoning to get that $\|z\|_{\mathcal{X}_T^0(0,L)} \leq C_1\|y_0\|_{H_\Gamma^3(0,L)}$, where C_1 only depends on T and $\|y_0\|_{L^2(0,L)}$. Thus, by carefully using the fact that $z = -p(x)y_{xxx} - y_x - yy_x$ (recall that p is piecewise constant), we obtain estimate (2.7). The details are easily fulfilled following [BSZ03]. □

2.3. **Adjoint system.** We now introduce the notion of weak solutions that will be used for our controllability problem in Section 3. The following definition is motivated by performing integration by parts as the ones done in Proposition 2.2.

Definition 2.7. For $(f, y_0) \in L^2(0,T;H^{-1}(0,L)) \times L^2(0,L)$ a function $y \in C([0,T], L^2(0,L))$ is called a weak solution of (2.3)-(TC) if it satisfies

$$\iint_Q y g dx dt + (y(T), \varphi_T)_{L^2(0,L)} = \int_0^T \langle f, \varphi \rangle_{H^{-1}(0,L)\times H_0^1(0,L)} dt + (y_0, \varphi(0))_{L^2(0,L)},$$

for all $(g, \varphi_T) \in L^1(0,T;L^2(0,L)) \times L^2(0,L)$, where φ is the mild solution of the adjoint system

$$\begin{cases} -\varphi_t - p(x)\varphi_{xxx} = g, & (t,x) \in (0,T) \times (0,L), \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_T(x), & x \in (0,L), \end{cases} \tag{2.8}$$

coupled by the corresponding transmission conditions (TC).

Remark 2.8. Similarly as we did in Proposition 2.3, we can state that for any $\varphi_T \in L^2(0,L)$, there is a unique mild solution φ of (2.8) which belongs to $\mathcal{X}_T^0(0,L)$. In particular, it also enjoys the Kato-type smoothing effect and henceforth the above definition makes sense.

Remark 2.9. A simple computation using integration by parts shows that a (piecewise) regular solution y of (2.3)-(TC) is also a solution in the above sense.

We now establish some regularity estimates for the adjoint system (2.8) that will be needed later in Section 4. We recall the definition of $H^s(0,L)$ given in (1.2) and we remark that in the next result we make the notational convention $H_\Gamma^{-1}(0,L) := H^{-1}(0,L)$, where H^{-1} is the usual dual space of H_0^1 equipped with the dual norm

$$\|\psi\|_{H^{-1}} = \sup_{\substack{h \in H_0^1(0,L) \\ \|h\|_{H_0^1} \leq 1}} \left| \int_0^L \psi h dx \right|$$

as a consequence of Riesz's representation theorem.

Proposition 2.10. *Let $T > 0$. If $\varphi_T \in \mathcal{D}(\mathcal{A}^*)$ and $g \in L^1(0,T;\mathcal{D}(\mathcal{A}^*))$, then there exists a unique strong solution φ of the adjoint equation (2.8) such that, for some $C = C(T, L, \Gamma, p) > 0$,*

$$\|\varphi\|_{C([0,T],H_\Gamma^3(0,L)) \cap L^2(0,T;H_\Gamma^4(0,L))} \leq C\big(\|\varphi_T\|_{H_\Gamma^3(0,L)} + \|g\|_{L^1(0,T;H_\Gamma^3(0,L))}\big). \tag{2.9}$$

Additionally, if $g \in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$, then for $s \in \{0,1,2,3\}$, we have

$$\|\varphi\|_{L^2(0,T;H_\Gamma^{s+1}(0,L))} \leq C\big(\|\varphi_T\|_{H_\Gamma^s(0,T)} + \|g\|_{L^2(0,T;H_\Gamma^{s-1}(0,L))}\big). \tag{2.10}$$

Proof. The existence follows by classical semigroup theory as done in Proposition 2.3, see [Paz83, Chapter 4]. We thus focus on obtaining estimates (2.10).

Step 1: case $s = 0$. This case is handled exactly as in Proposition 2.3 by employing Proposition 2.2. Moreover, arguing as in the proof of Proposition 2.3, but instead by choosing the multiplier $q_{N-1}(x) = (x-L)/\sqrt{p_{N-1}}$ for $x \in I_{N-1}$ and $q_k(x) = (x - a_{k+1})/\sqrt{p_{k+1}} + q_{k+1}(a_{k+1}^+)$ for $x \in I_k$ for all $k \in [\![0, N-2]\!]$, we obtain

$$3\int_0^T \int_0^L \sqrt{p}|\varphi_x|^2 dx dt + \int_0^L |q\varphi(0,x)|^2 dx = \int_0^L |q\varphi_T(x)|^2 dx + 2\int_0^T \int_0^L |q|\varphi g dx dt.$$

Thus, since p is bounded by below, by using Poincaré's inequality, for any $\varepsilon > 0$ we have

$$\|\varphi_x\|_{L^2(L^2)}^2 \leq C\|\varphi_T\|_{L^2}^2 + \varepsilon\|\varphi_x\|_{L^2(L^2)}^2 + C_\varepsilon\|g\|_{L^2(H^{-1})}^2.$$

By choosing $\varepsilon > 0$ small enough, we readily get inequality (2.10) for $s = 0$.

Step 2: case $s = 3$. Let $\varphi_T \in \mathcal{D}(\mathcal{A}^*)$ and $g \in L^1(0, T; \mathcal{D}(\mathcal{A}^*))$. By classical semigroup theory, then $\varphi \in C([0, T], \mathcal{D}(\mathcal{A}^*))$. Hence, if we let $w = \mathcal{A}^*\varphi$, it is a mild solution of (2.8) with initial data $\mathcal{A}^*\varphi_T$ and source term $\mathcal{A}^*g \in L^1(0, T; L^2(0, L))$. Furthermore, we can perform the same analysis as in the case $s = 0$, which lead us to

$$\|w\|_{L^2(0,T;H^1(0,L))} \leq C\big(\|\mathcal{A}^*\varphi_T\|^2_{L^2(0,L)} + \|\mathcal{A}^*g\|^2_{L^2(H^{-1})}\big). \tag{2.11}$$

To estimate the last term in the right-hand side above, we use

$$\|\mathcal{A}^*g\|_{H^{-1}} = \sup_{\substack{h\in H^1_0(0,L)\\ \|h\|_{H^1_0}\leq 1}} \left|\int_0^L p g_{xxx} h dx\right| = \sup_{\substack{h\in H^1_0(0,L)\\ \|h\|_{H^1_0}\leq 1}} \left|\int_0^L p g_{xx} h_x dx\right|.$$

Indeed, as we did in Lemma 2.1, we have $\mathcal{A}^*g \in L^2(0, L)$ and thus $\langle \mathcal{A}^*g, h\rangle_{L^2(0,L)} = -\langle pg_{xx}, h_x\rangle_{L^2(0,L)}$ due to the transmission conditions. By Cauchy-Schwarz's inequality, we get

$$\|\mathcal{A}^*g\|_{H^{-1}} = \sup_{\substack{h\in H^1_0(0,L)\\ \|h\|_{H^1_0}\leq 1}} \left|\int_0^L p g_{xx} h_x dx\right| \leq \|pg_{xx}\|_{L^2} \leq C\|g\|_{H^2_\Gamma}. \tag{2.12}$$

As p is bounded by below, by using the transmission conditions again, we have that

$$\|\varphi\|_{L^2(0,T;H^4_\Gamma(0,L))} \lesssim \|w\|_{L^2(0,T;H^1(0,L))} + \|\varphi\|_{L^2(0,T;H^1(0,L))} \tag{2.13}$$

and gathering inequalities (2.11)-(2.12)-(2.13), we have

$$\|\varphi\|_{L^2(0,T;H^4_\Gamma(0,L))} \lesssim \|\varphi_T\|_{H^3_\Gamma} + \|g\|_{L^2(0,T;H^2_\Gamma(0,L))},$$

proving estimate (2.10) for $s = 3$.

Step 3: cases $s = 1, 2$. Observe that with a similar reasoning we can obtain estimates for $s = 1$ and $s = 2$. Indeed, let us introduce

$$v(t) := e^{t\mathcal{A}^*}(\sqrt{p}\partial_x^2\varphi_T) + \int_0^t e^{(t-s)\mathcal{A}^*}(\sqrt{p}\partial_x^2 g)(s)ds,\ t \in [0, T].$$

Note that v is a mild solution on $\mathcal{X}^0_T(0, L)$ of (2.8) with initial condition $\sqrt{p}\partial_x^2\varphi_T \in L^2(0, L)$ and source term $\sqrt{p}\partial_x^2 g \in L^1(0, T; L^2(0, L))$. Thus

$$\|v\|_{L^2(0,T;H^1(0,L))} \lesssim \|\sqrt{p}\partial_x^2\varphi_T\|_{L^2(0,L)} + \|\sqrt{p}\partial_x^2 g\|_{L^2(H^{-1})}. \tag{2.14}$$

As in the previous case, integrating by parts and using that the transmission conditions, we have

$$\|\sqrt{p}\partial_x^2 g\|_{L^2(H^{-1})} = \sup_{\substack{h\in H^1_0(0,L)\\ \|h\|_{H^1_0}\leq 1}} \left|\int_0^L \sqrt{p} g_{xx} h dx\right| = \sup_{\substack{h\in H^1_0(0,L)\\ \|h\|_{H^1_0}\leq 1}} \left|\int_0^L \sqrt{p} g_x h_x dx\right| \leq \|\sqrt{p} g_x\|_{L^2}. \tag{2.15}$$

Since p is piecewise constant, we see that $v = \sqrt{p}\varphi$ on $(0, T) \times (a_k, a_{k+1})$ for $k \in [\![0, N-1]\!]$. With this observation and using that p is bounded by above and below, putting together inequalities (2.14) and (2.15), we arrive to

$$\|\varphi\|_{L^2(0,T;H^3_\Gamma(0,L))} \lesssim \|\varphi_T\|_{H^2_\Gamma(0,L)} + \|g\|_{L^2(0,T;H^1(0,L))}.$$

Lastly, note that

$$u(t) := e^{t\mathcal{A}^*}(\partial_x\varphi_T) + \int_0^t e^{(t-s)\mathcal{A}^*}(\partial_x g)(s)ds,\ t \in [0, T],$$

is a mild solution of (2.3) in $\mathcal{X}_T^0(0,L)$ with initial data $\partial_x\varphi_T \in L^2(0,L)$ and source term $\partial_x g \in L^1(0,T;L^2(0,L))$. With the aid of Duhamel's formula we get $\partial_x\varphi = u$ in $\mathcal{X}_T^0(0,L)$ and the estimate for $s=1$ follows as in the previous cases. □

Remark 2.11. Estimates like (2.10) are usually obtained by means of interpolation theory. However, we avoid such kind of arguments since the description of the interpolation space $[L^2(0,L),\mathcal{D}(\mathcal{A}^*)]_\theta$ is a delicate issue. Up to the author's knowledge, it is not a result available in the literature and it is outside of the scope of the present article.

We now establish a similar result for the adjont system to the linearized version of (1.7) with a regular source term. These estimates are key for the proof of Proposition 4.1.

Proposition 2.12. *Let $T>0$ be given and assume $\overline{y}\in\mathcal{X}_T^0(0,L)$. Then for any $\varphi_T\in L^2(0,L)$ and $g\in L^1(0,T;L^2(0,L))$, there exists a unique solution $\varphi\in\mathcal{X}_T^0(0,L)$ of*

$$\begin{cases} -\varphi_t - p(x)\varphi_{xxx} - \varphi_x - \overline{y}\varphi_x = g, & (t,x)\in(0,T)\times(0,L),\\ \varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0, & t\in(0,T),\\ \varphi(T,x)=\varphi_T(x), & x\in(0,L).\end{cases}$$

Additionally, if $\overline{y}\in\mathcal{X}^3_{\Gamma,T}(0,L)$, $\varphi_T\in\mathcal{D}(\mathcal{A}^*)$ and $g\in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$, then for $s\in\{0,1,2,3\}$, we have

$$\|\varphi\|_{L^2(0,T;H^{s+1}_\Gamma(0,L))} \le C\big(\|\varphi_T\|_{H^s_\Gamma(0,T)} + \|g\|_{L^2(0,T;H^{s-1}_\Gamma(0,L))}\big) \tag{2.16}$$

Proof. By linearity, we split $\varphi=\varphi^1+\varphi^2$ where φ^1 solves the system with initial data φ_T and source term g, and φ^2 solves the system with potential $\overline{y}$. To treat φ^1 we use Proposition 2.10 and to treat φ^2, we use a fixed point argument following the same steps of the proof of Proposition 2.6. The estimates follow from Proposition 2.10. We omit the details. □

Remark 2.13. Note that the statement of Proposition 2.12 is not empty, as shown by Proposition 2.6.

3. A global Carleman estimate

We first introduce a weight function with internal observation. Let $j\in\{0,\dots,N-1\}$ be fixed in the sequel and let $\omega_0\Subset I_j$.

Lemma 3.1. *There exists a continuous function $\beta\in C([0,L])$ such that $\beta|_{\overline{I_k}}\in C^3(\overline{I_k})$ for $k\in[\![0,N-1]\!]$, satisfying the following properties*

(1) for some $r>0$, it holds that

$$\min_{x\in[0,L]}\beta\ge r \quad \text{and} \quad \beta_x\neq 0 \text{ in } \overline{I_j}\setminus\omega_0$$

and depending on the value of j:

(a) if $j\notin\{0,N-1\}$ then

$$\begin{cases} \beta_x\ge r>0 & \text{in } \overline{I_k} \text{ for } k\in[\![0,j-1]\!],\\ \beta_x\le -r<0 & \text{in } \overline{I_k} \text{ for } k\in[\![j+1,N-1]\!];\end{cases}$$

(b) if $j=0$ then

$$\beta_x\le -r<0 \text{ in } \overline{I_k} \text{ for } k\in[\![0,N-1]\!];$$

(c) if $j=N-1$ then

$$\beta_x\ge r>0 \text{ in } \overline{I_k} \text{ for } k\in[\![0,N-1]\!];$$

(2) for some $\kappa\in(1,2)$ it holds that

$$\kappa\max_{x\in\overline{I_k}}\beta < 2\min_{x\in\overline{I_k}}\beta,\ \ k\in[\![0,N-1]\!]; \tag{3.1}$$

(3) at the interface the following transmission conditions holds:

$$\begin{cases} \beta(a_k^-) &= \beta(a_k^+), & k \in [\![1, N-1]\!], \\ \sqrt{p_{k-1}}\beta_x(a_k^-) &= \sqrt{p_k}\beta_x(a_k^+), & k \in [\![1, N-1]\!], \\ p_{k-1}\beta_{xx}(a_k^-) &= p_k\beta_{xx}(a_k^+), & k \in [\![1, N-1]\!]. \end{cases} \tag{3.2}$$

Proof. We will show the existence of such a weight-function β by explicitly constructing a piecewise linear function, excepting the observation interval I_j, satisfying all the afore given properties. Although this is enough to obtain our Carleman estimate, we point out that a more general β could be constructed, see for instance [BDLR07, Lemma 1.1, Lemma 2.1].

In what follows, we will denote the restriction of β to I_k by $\beta_{|_{I_k}} := \beta_k$ for $k \in [\![0, N-1]\!]$.

Step 1. Piecewise affine on the left. For $k \in [\![0, j-1]\!]$ we define $\beta_k(x) = m_k(x - a_k) + c_k$, where m_k and c_k are to be chosen. The third equation of (3.2) is trivially satisfied. By the second condition, we see that

$$\sqrt{p_k}m_k = \sqrt{p_{k+1}}m_{k+1}, \ k \in [\![0, j-1]\!].$$

By taking $m_0 > 0$, we inductively obtain $\{m_1, \dots, m_j\}$, with each m_k depending on m_0 and p_i for $0 \le i \le k$. Furthermore, we ensure that β' is positive and bounded by below on each one of these intervals.

Step 2. Quartic polynomial on the observation zone. To start the construction, let us impose $\beta_j''(x) = m_j(x - a_j)(x - a_{j+1})$, for some $m_j \in \mathbb{R}$ to be determined. With this choice, we ensure that the third equation of (3.2) is satisfied on both sides of I_j. We thus have

$$\beta_j'(x) = m_j \int_{a_j}^{x} (t - a_j)(t - a_{j+1})dt + n_j, \tag{3.3}$$

where $n_j := \frac{\sqrt{p_{j-1}}m_{j-1}}{\sqrt{p_j}}$, prescribed by the transmission condition on the left. For the condition on the right, we see that our choice of m_j will determine $\beta_j'(a_{j+1}^-)$ and thus we choose it so that $\beta_j'(a_{j+1}^-) < 0$. To ensure the latter condition, a simple computation leads us to impose impose

$$\frac{6n_j}{(a_{j+1} - a_j)^3} < m_j. \tag{3.4}$$

Let us suppose that the observation set is given by $\omega_0 = (a_j + \delta_0, a_{j+1} - \delta_1)$ for $\delta_0, \delta_1 > 0$. Let us denote by $I_j = I_j(x)$ the integral term on (3.3), defined for $x \in [a_j, a_{j+1}]$. Observe that I_j is negative and strictly decreasing. To ensure that the conditions $\beta_j'(a_j + \delta_0) > 0$ and $\beta_j'(a_{j+1} - \delta_1) < 0$ hold, along with (3.4), we impose that m_j satisfies

$$\max\left\{\frac{6n_j}{(a_{j+1} - a_j)^3}, \frac{6n_j}{|I_j(a_{j+1} - \delta_1)|}\right\} < m_j < \frac{6n_j}{|I_j(a_j + \delta_0)|},$$

which is consistent since $|I_j|$ is strictly increasing. With these choices, by the intermediate value theorem we ensure that any zero of β_j' must lie inside of ω_0 and thus $\beta_j' \neq 0$ in $\overline{I_j} \setminus \omega_0$.

Step 3. Piecewise affine on the right. As in the first step, for $k \in [\![j+1, N-1]\!]$ we define $\beta_k(x) = m_k(x - a_k) + c_k$, where m_k and c_k are to be chosen. Once imposed the condition $\sqrt{p_j}\beta_j'(a_{j+1}^-) = \sqrt{p_{j+1}}m_{j+1}$, the coefficient m_{j+1} is determined by our previous choices of m_0 and m_j, along with the parameters $p_0, \dots, p_{j+1}$ and $a_1, \dots, a_{j+1}$. We then define the remaining m_k's inductively by using the second equation in (3.2).

Step 4. Final bounds. We start by asking $c_0 > 0$ to be large enough so that $\beta_0 > 0$ and by monotony this implies that $\beta_k > 0$ for each $k \in [\![1, j-1]\!]$. Furthermore, by choosing $c_0 > \frac{\kappa}{2}(a_{j+1} - a_j)$ we verify (3.1) on $\overline{I_0}$. By imposing the continuity condition, inductively, we obtain

$$c_k = \sum_{i=1}^{k} m_i(a_i - a_{i-1}) + c_0,\ k \in [\![1, j-1]\!]. \tag{3.5}$$

Once again, inductively, we see that by taking c_0 large enough satisfying

$$c_0 > \frac{\kappa}{(2-\kappa)} m_{k-1}(a_k - a_{k-1}) - \sum_{i=0}^{k-1} m_i(a_i - a_{i-1}),\ k \in [\![2, j-1]\!],$$

we verify property (3.1) for $k \in [\![0, j-1]\!]$. On I_j we have

$$\beta_j(x) := \int_{a_j}^{x} \beta_j'(s)ds + c_j,\ x \in I_j.$$

The continuity on the left-end point of the interval extends the validity of (3.5) for $k = j$ as well and on the right-end point we get $c_{j+1} = \beta_j(a_{j+1}^-)$. If $x^*, x_* \in \overline{I_j}$ denote the points where β_j attains its maximum and minimum, respectively, since c_j depends in an affine way with respect to c_0, we further impose that c_0 is such that

$$\kappa \int_0^{x^*} \beta_j'(s)ds - \int_0^{x_*} \beta_j'(s)ds < (2-\kappa)c_j,$$

and this inequality implies that (3.1) is verified for $k = j$. In the remaining pieces, we proceed as before: once imposing the continuity condition at a_{j+1}, we see that c_{j+1}, and hence c_k for $k \in [\![j+1, N-1]\!]$, depends in an affine way with respect to c_0, so we can verify (3.1) for $k \in [\![j+1, N-1]\!]$ by choosing c_0 large enough.

By piecing together all the β_k's we obtain the desired weight function β. □

Remark 3.2. Since we are deriving a two-parameter Carleman estimate, a condition on the second derivative of the weight function is not required, unlike in the one-parameter case. However, we retain the second-order transmission condition, as it simplifies some computations later on.

Let us set $Q' = (0,T) \times ((0,L) \setminus \Gamma)$. The weight functions (1.5) satisfy the following identities

$$\begin{aligned} &\partial_x \eta = -\lambda \beta' \xi, && \partial_x \xi = \lambda \beta' \xi, && \text{in } Q', \\ &\partial_t \eta = \frac{2t - T}{t(T-t)} \eta, && \partial_t \xi = \frac{2t-T}{t(T-t)} \xi, && \text{in } Q. \end{aligned}$$

We have that for each $k \in \mathbb{N}$, there exists $C > 0$ independent of $\lambda > 0$ such that

$$|\partial_x^k \eta(t,x)| + |\partial_x^k \xi(t,x)| \le C(\lambda^k + \lambda^{k-1} + \ldots + \lambda + 1)\xi(t,x),\ (t,x) \in Q'. \tag{3.6}$$

Furthermore, due to the properties of β, there exists a constant $C = C(T) > 0$ such that

$$C^{-1} \le \xi(t,x) \ \text{ and } \ |\partial_t \eta(t,x)| + |\partial_t \xi(t,x)| \le C\xi^2(t,x),\ (t,x) \in Q. \tag{3.7}$$

These estimates will give us the heuristics to identify the dominating and lower order terms coming from the integration by parts later on.

3.1. **Proof of Theorem 1.1.** Let $s>0$ and define $\mathcal{V}_s=\{e^{-s\eta}u : u\in\mathcal{V}\}$. For $u\in\mathcal{V}$ set $w=e^{-s\eta}u$ and introduce the conjugate operator

$$\mathcal{L}_\eta w = e^{-s\eta}\mathcal{L}(e^{s\eta}w)=(\mathcal{L}_1+\mathcal{L}_2+\mathcal{R})w$$

where

$$\begin{aligned}\mathcal{L}_1 w &= w_t+3ps^2\eta_x^2 w_x+pw_{xxx}+3pms^2\eta_x\eta_{xx}w,\\ \mathcal{L}_2 w &= ps^3\eta_x^3 w+3ps\eta_x w_{xx}+3sw_x(p\eta_x)_x,\end{aligned}$$

and

$$\mathcal{R}w=bs\eta_x w+bw_x+ps\eta_{xxx}w+3ps^2\eta_x\eta_{xx}w+dw+s\eta_t w-3sp_x\eta_x w_x-3pms^2\eta_x\eta_{xx}w,$$

for some constant $m>0$, to be chosen later. By Lemma 3.1, since β satisfies the transmission conditions, the conjugate function satisfies them as well:

$$\begin{cases} w(t,a_k^-) &= w(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!],\\ \sqrt{p_{k-1}}w_x(t,a_k^-) &= \sqrt{p_k}w_x(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!],\\ p_{k-1}w_{xx}(t,a_k^-) &= p_k w_{xx}(t,a_k^+), & t\in(0,T),\ k\in[\![1,N-1]\!].\end{cases}$$

Taking the L^2-norm to $\mathcal{L}_\eta$ we obtain

$$\|\mathcal{L}_1 w\|^2_{L^2(Q)}+\|\mathcal{L}_2 w\|^2_{L^2(Q)}+2\langle\mathcal{L}_1 w,\mathcal{L}_2 w\rangle_{L^2(Q)}=\|\mathcal{L}_\eta w-\mathcal{R}w\|^2_{L^2(Q)}$$

and then

$$\|\mathcal{L}_1 w\|^2_{L^2(Q)}+\|\mathcal{L}_2 w\|^2_{L^2(Q)}+2\langle\mathcal{L}_1 w,\mathcal{L}_2 w\rangle_{L^2(Q)}\le 2\|\mathcal{L}_\eta w\|^2_{L^2(Q)}+2\|\mathcal{R}w\|^2_{L^2(Q)}.$$

3.1.1. *Double product term.* To fix notation, in what follows the symbol $|_0^L$ denotes the evaluation at the end points considering the interface, namely, $\mu|_0^L:=\sum_{k=0}^{N-1}\mu|_{a_k}^{a_{k+1}}$. Henceforth, we will write

$$\mu\Big|_0^L=\mu(L)-\mu(0)-\sum_{a\in\Gamma}[\mu]_a.$$

Denote by I_{ij} for $i\in[\![1,4]\!]$, $j\in[\![1,3]\!]$ the $ij-$term of the L^2-product $\langle\mathcal{L}_1 w,\mathcal{L}_2 w\rangle_{L^2(Q)}$. We have that $w(0,t)=w(L,t)=0$ for all $t\in(0,T)$ and $w(x,0)=w(x,T)=0$ for all $x\in(0,L)$. In what follows, for each term I_{ij} we will perform several integration by parts and we will write once explicitly all the terms. Then we will gather these terms into two different groups: the distributed and boundary-interface terms. Within these groups we will split the terms with respect to the powers of s, λ and ξ into dominating and lower-order terms: the former will give produce the weighted norms we are looking for and the latter will be absorbed for large s and λ.

We perform the integration by parts below:

$$I_{11}=\iint_Q ps^3\eta_x^3 ww_t dxdt=\frac{3}{2}s^3\lambda^3\iint_Q p\beta_x^3\xi^2\xi_t|w|^2dxdt,$$

$$\begin{aligned}I_{12}&=3s\iint_Q p\eta_x w_{xx}w_t dxdt\\ &=-I_{13}-\frac{3}{2}s\lambda\iint_Q p\beta_x\xi_t|w_x|^2dxdt-3s\lambda\int_0^T p\beta_x\xi w_t w_x dt\Big|_0^L,\end{aligned}$$

$$\begin{aligned}I_{21}&=3s^5\iint_Q p^2\eta_x^5 ww_x dxdt\\ &=\frac{3}{2}s^5\lambda^5\iint_Q(p^5\beta_x^5)_x\xi^5|w|^2dxdt+\frac{15}{2}s^5\lambda^6\iint_Q p^2\beta_x^6\xi^5|w|^2dxdt-\frac{3}{2}s^5\lambda^5\int_0^T p^2\beta_x^5\xi^5|w|^2dt\Big|_0^L,\end{aligned}$$

$$\begin{aligned}
I_{22} &= 9s^3 \iint_Q p^2 \eta_x^3 w_x w_{xx} dxdt \\
&= \frac{9}{2} s^3 \lambda^3 \iint_Q (p^2 \beta_x^3)_x \xi^3 |w_x|^2 dxdt + \frac{27}{2} s^3 \lambda^4 \iint_Q p^2 \beta_x^4 \xi^3 |w_x|^2 dxdt - \frac{9}{2} s^3 \lambda^3 \int_0^T p^2 \beta_x^3 \xi^3 |w_x|^2 dt \Big|_0^L,
\end{aligned}$$

$$\begin{aligned}
I_{23} &= 9 \iint_Q p s^3 \eta_x^2 (p\eta_x)_x |w_x|^2 dxdt \\
&= -9 s^3 \lambda^3 \iint_Q p \beta_x^2 (p\beta_x)_x \xi^3 |w_x|^2 dxdt - 9 s^3 \lambda^4 \iint_Q p^2 \beta_x^4 \xi^3 |w_x|^2 dxdt,
\end{aligned}$$

$$\begin{aligned}
I_{31} &= s^3 \iint_Q p^2 \eta_x^3 w w_{xxx} dxdt \\
&= \frac{s^3\lambda^3}{2} \iint_Q (p^2 \beta_x^3 \xi^3)_{xxx} |w|^2 dxdt - \frac{3}{2} s^3 \lambda^3 \iint_Q (p^2 \beta_x^3)_x \xi^3 |w_x|^2 dxdt - \frac{9}{2} s^3 \lambda^4 \iint_Q p^2 \beta_x^4 \xi^3 |w_x|^2 dxdt \\
&\quad - \frac{s^3\lambda^3}{2} \int_0^T (p^2 \beta_x^3 \xi^3)_{xx} |w|^2 dt \Big|_0^L + s^3 \lambda^3 \int_0^T (p^2 \beta_x^3 \xi^3)_x w w_x dt \Big|_0^L + \frac{s^3\lambda^3}{2} \int_0^T p^2 \beta_x^3 \xi^3 |w_x|^2 dt \Big|_0^L \\
&\quad - s^3 \lambda^3 \int_0^T p^2 \beta_x^3 \xi^3 w w_{xx} dt \Big|_0^L,
\end{aligned}$$

$$\begin{aligned}
I_{32} &= 3s \iint_Q p^2 \eta_x w_{xx} w_{xxx} dxdt \\
&= \frac{3}{2} s \lambda \iint_Q (p^2 \beta_x)_x \xi |w_{xx}|^2 dxdt + \frac{3}{2} s \lambda^2 \iint_Q p^2 \beta_x^2 \xi |w_{xx}|^2 dxdt - \frac{3}{2} s \lambda \int_0^T p^2 \beta_x \xi |w_{xx}|^2 dt \Big|_0^L,
\end{aligned}$$

$$\begin{aligned}
I_{33} &= 3s \iint_Q p (p\eta_x)_x w_x w_{xxx} dxdt \\
&= -\frac{3}{2} s \lambda \iint_Q (p(p\beta_x \xi)_x)_{xx} |w_x|^2 + 3 s \lambda \iint_Q p (p\beta_x)_x \xi |w_{xx}|^2 dxdt + 3 s \lambda^2 \iint_Q p^2 \beta_x^2 \xi |w_{xx}|^2 dxdt \\
&\quad + \frac{3}{2} s \lambda \int_0^T (p(p\beta_x \xi)_x)_x |w_x|^2 dt \Big|_0^L - 3 s \lambda \int_0^T p (p\beta_x \xi)_x w_x w_{xx} dt \Big|_0^L,
\end{aligned}$$

$$\begin{aligned}
I_{41} &= 3 m s^5 \iint_Q p^2 \eta_x^4 \eta_{xx} |w|^2 dxdt \\
&= -3 m s^5 \lambda^5 \iint_Q p^2 \beta_x^4 \beta_{xx} \xi^5 |w|^2 dxdt - 3 m s^5 \lambda^6 \iint_Q p^2 \beta_x^6 \xi^6 |w|^2 dxdt,
\end{aligned}$$

$$I_{42} = 9 m s^3 \iint_Q p^2 \eta_x^2 \eta_{xx} w w_{xx} dxdt$$

$$
\begin{aligned}
= & -\frac{9}{2}ms^3\lambda^3 \iint_Q (p^2\beta_x^2\beta_{xx}\xi^3)_{xx}|w|^2dxdt - \frac{9}{2}ms^3\lambda^4 \iint_Q (p^2\beta_x^4\xi^3)_{xx}|w|^2dxdt \\
& + 9ms^3\lambda^3 \iint_Q p^2\beta_x^2\beta_{xx}\xi^3|w_x|^2dxdt + 9ms^3\lambda^4 \iint_Q p^2\beta_x^4\xi^3|w_x|^2dxdt \\
& + \frac{9}{2}ms^3\lambda^3 \int_0^T (p^2\beta_x^2\beta_{xx}\xi^3)_x|w|^2dt\Big|_0^L + \frac{9}{2}ms^3\lambda^4 \int_0^T (p^2\beta_x^3\xi^3)_x|w|^2dt\Big|_0^L \\
& - 9ms^3\lambda^3 \int_0^T p^2\beta_x^2\beta_{xx}\xi^3ww_xdt\Big|_0^L - 9ms^3\lambda^4 \int_0^T p^2\beta_x^4\xi^3ww_xdt\Big|_0^L,
\end{aligned}
$$

$$
\begin{aligned}
I_{43} &= 9ms^3 \iint_Q p(p\eta_x)_x\eta_x\eta_{xx}ww_xdxdt \\
&= \frac{9}{2}ms^3\lambda^3 \iint_Q [p(p\beta_x\xi)_x\beta_x\beta_{xx}\xi^2]_x|w|^2dxdt + \frac{9}{2}ms^3\lambda^4 \iint_Q [p(p\beta_x\xi)_x\beta_x^3\xi^2]_x|w|^2dxdt + \\
&\quad - \frac{9}{2}ms^3\lambda^3 \int_0^T p(p\beta_x\xi)_x\beta_x\beta_{xx}\xi^2|w|^2dt\Big|_0^L - \frac{9}{2}ms^3\lambda^4 \int_0^T p(p\beta_x\xi)_x\beta_x^3\xi^2|w|^2dt\Big|_0^L.
\end{aligned}
$$

3.1.2. *Gathering terms.* We split the double product terms as follows

$$
\begin{aligned}
\langle \mathcal{L}_1 w, \mathcal{L}_2 w\rangle_{L^2(Q)} = & \left(\frac{15}{2} - 3m\right) s^5\lambda^6 \iint_Q p^2\beta_x^6\xi^5|w|^2dxdt \\
& + 9ms^3\lambda^4 \iint_Q p^2\beta_x^4\xi^3|w_x|^2dxdt + \frac{9}{2}s\lambda^2 \iint_Q p^2\beta_x^2\xi|w_{xx}|^2dxdt + \mathfrak{D}^{low} + \mathfrak{B}
\end{aligned}
$$

where $\mathfrak{D}^{low}$ and $\mathfrak{B}$ gather the lower order distributed and boundary-interface terms, respectively. To be precise, using inequalities (3.6) and (3.7) we have the following estimate

$$
\begin{aligned}
|\mathfrak{D}^{low}| \lesssim & (s^5\lambda^5 + s^3\lambda^6) \iint_Q \xi^5|w|^2dxdt \\
& + (s^3\lambda^3 + s\lambda^4) \iint_Q \xi^3|w_x|^2dxdt + s\lambda \iint_Q \xi|w_{xx}|^2dxdt. \quad (3.8)
\end{aligned}
$$

For the boundary-interface terms, without taking into account any of the boundary conditions nor the transmission conditions, we have

$$
\begin{aligned}
\mathfrak{B} = & -\frac{3}{2}s^5\lambda^5 \int_0^T p^2\beta_x^5\xi^5|w|^2dt\Big|_0^L + \frac{9}{2}ms^3\lambda^4 \int_0^T (p^2\beta_x^3\xi^3)_x|w|^2dt\Big|_0^L \\
& - \frac{9}{2}ms^3\lambda^4 \int_0^T p(p\beta_x\xi)_x\beta_x^3\xi^3|w|^2dt\Big|_0^L - \frac{s^3\lambda^3}{2} \int_0^T (p^2\beta_x^3\xi^3)_{xx}|w|^2dt\Big|_0^L \\
& + \frac{9}{2}ms^3\lambda^3 \int_0^T (p^2\beta_x^2\beta_{xx}\xi^3)_x|w|^2dt\Big|_0^L - \frac{9}{2}ms^3\lambda^3 \int_0^T p(p\beta_x\xi)_x\beta_x\beta_{xx}\xi^2|w|^2dt\Big|_0^L \\
& - 4s^3\lambda^3 \int_0^T p^2\beta_x^3\xi^3|w_x|^2dt\Big|_0^L + \frac{3}{2}s\lambda \int_0^T [p(p\beta_x\xi)_x]_x|w_x|^2dt\Big|_0^L - \frac{3}{2}s\lambda \int_0^T p^2\beta_x\xi|w_{xx}|^2dt\Big|_0^L \\
& - 3s\lambda \int_0^T p(p\beta_x\xi)_xw_xw_{xx}dt\Big|_0^L - s^3\lambda^3 \int_0^T p^2\beta_x^3\xi^3ww_{xx}dt\Big|_0^L + s^3\lambda^3 \int_0^T (p^2\beta_x^3\xi^3)_xww_xdt\Big|_0^L
\end{aligned}
$$

$$- 9ms^3\lambda^3 \int_0^T p^2\beta_x^2\beta_{xx}\xi^3 ww_x dt\Big|_0^L - 9ms^3\lambda^4 \int_0^T p^2\beta_x^4\xi^3 ww_x dt\Big|_0^L - 3s\lambda \int_0^T p\beta_x\xi w_t w_x dt\Big|_0^L .$$

In view of the boundary and the transmission conditions, we split $\mathfrak{B}$ as follows

$$\mathfrak{B} = \mathfrak{B}_L + \mathfrak{B}_0 + \mathfrak{B}^* + \mathfrak{B}_\Gamma,$$

where each one of these terms is described below. First of all, $\mathfrak{B}_L$ and $\mathfrak{B}_0$ correspond to those terms at $x = L$ and 0, respectively, that have fixed sign

$$\mathfrak{B}_L = -4s^3\lambda^3 \int_0^T p(L)^2\beta_x^3(L)\xi^3(t,L)|w_x(t,L)|^2 dt - \frac{3}{2}s\lambda \int_0^T p(L)^2\beta_x(L)\xi(t,L)|w_{xx}(t,L)|^2 dt,$$

$$\mathfrak{B}_0 = 4s^3\lambda^3 \int_0^T p(0)^2\beta_x^3(0)\xi^3(t,0)|w_x(t,0)|^2 dt + \frac{3}{2}s\lambda \int_0^T p(0)^2\beta_x(0)\xi(t,0)|w_{xx}(t,0)|^2 dt,$$

and $\mathfrak{B}^*$ corresponds to the terms without fixed sign at the boundary

$$\mathfrak{B}^* = \frac{3}{2}s\lambda \int_0^T (p(p\beta_x\xi)_x)_x\big|_{x=L}|w_x(t,L)|^2 dt - \frac{3}{2}s\lambda \int_0^T (p(p\beta_x\xi)_x)_x\big|_{x=0}|w_x(t,0)|^2 dt$$
$$+ 3s\lambda \int_0^T (p(p\beta_x\xi)_x)\big|_{x=0} w_x(t,0)w_{xx}(t,0)dt - 3s\lambda \int_0^T (p(p\beta_x\xi)_x)\big|_{x=L} w_x(t,L)w_{xx}(t,L)dt.$$

In the same spirit as before, we split the terms at the interface as follows

$$\mathfrak{B}_\Gamma = \mathfrak{B}_\Gamma^{dom} + \mathfrak{B}_\Gamma^{low} = \sum_{a\in\Gamma} \big(\mathfrak{B}_\Gamma^{dom}(a) + \mathfrak{B}_\Gamma^{low}(a)\big).$$

Here

$$\mathfrak{B}_\Gamma^{dom}(a) = \frac{3}{2}s^5\lambda^5 \int_0^T [\beta_x^5 p^2]_a \xi^5(t,a)|w(t,a)|^2 dt + 4s^3\lambda^3 \int_0^T [p^2\beta_x^3|w_x|^2]_a \xi^3(t,a)dt$$
$$+ s^3\lambda^3 \int_0^T [p^2\beta_x^3\xi^3 ww_{xx}]_a dt + \frac{3}{2}s\lambda \int_0^T [p^2\beta_x|w_{xx}|^2]_a \xi(t,a)dt$$
$$+ 3s\lambda \int_0^T [p\beta_x\xi w_t w_x]_a dt$$

and $\mathfrak{B}_\Gamma^{low}$ gathers the remaining terms at the interface, which will be shown to be of lower order with respect to $\mathfrak{B}_\Gamma^{dom}$.

3.1.3. *Estimates for the distributed terms.* First we fix $m \in (0, 5/2)$. Henceforth the generic constant depends on L, T, ρ_0, ρ_1, $\|\beta\|_{C^3}$, r, s_0 and λ_0, where s_0 and λ_0 will be chosen later. The dominating terms are bounded by below as follows

$$\left(\frac{15}{2} - 3m\right) s^5\lambda^6 \iint_Q p^2\beta_x^6\xi^5|w|^2 dxdt + 9ms^3\lambda^4 \iint_Q p^2\beta_x^4\xi^3|w_x|^2 dxdt + \frac{9}{2}s\lambda^2 \iint_Q p^2\beta_x^2\xi|w_{xx}|^2 dxdt$$
$$\gtrsim \int_0^T \int_{(0,L)\setminus\omega_0} (s^5\lambda^6\xi^5|w|^2 dxdt + s^3\lambda^4\xi^3|w_x|^2 dxdt + s\lambda^2\xi|w_{xx}|^2)dxdt.$$

Let us introduce the following weighted norm

$$\|w\|_{s,\lambda,\xi}^2 := s^5\lambda^6 \iint_Q \xi^5|w|^2 dxdt + s^3\lambda^4 \iint_Q \xi^3|w_x|^2 dxdt + s\lambda^2 \iint_Q \xi|w_{xx}|^2 dxdt.$$

Thus, estimate (3.8) reads as

$$|\mathfrak{D}^{low}| \lesssim \left(\frac{1}{s^2}+\frac{1}{\lambda}\right)\|w\|^2_{s,\lambda,\xi}.$$

and in consequence, by choosing s_0 and λ_0 large enough, we get

$$\begin{aligned}\|w\|^2_{s,\lambda,\xi}+\mathfrak{B} &\lesssim \langle \mathcal{L}_1 w, \mathcal{L}_2 w\rangle_{L^2(Q)} \\ &+\iint_{(0,T)\times\omega_0}(s^5\lambda^6\xi^5|w|^2dxdt+s^3\lambda^4\xi^3|w_x|^2dxdt+s\lambda^2\xi|w_{xx}|^2)dxdt, \quad (3.9)\end{aligned}$$

for all $s\geq s_0$ and $\lambda\geq\lambda_0$. Additionally, for the definition of $\mathcal{R}$, we observes that the highest powers are $s^4\lambda^6$ for the zero order term and $s^2\lambda^2$ for the first order term. From the regularity assumptions of p, b and d we obtain we observe that for the residue term one has the estimate

$$\|\mathcal{R}w\|^2_{L^2(Q)} \lesssim s^4\lambda^6\iint_Q \xi^4|w|^2dxdt + s^2\lambda^2\iint_Q\xi^2|w_x|^2dxdt \lesssim \left(\frac{1}{s}+\frac{1}{s\lambda^2}\right)\|w\|^2_{s,\lambda,\xi}.$$

3.1.4. *Estimates for the boundary terms.* From the properties of β given by Lemma 3.1, we have both $\mathfrak{B}_0>0$ and $\mathfrak{B}_L>0$, since $\beta_x(0)$ and $-\beta_x(L)$ are both positively bounded by below. To treat the terms contained in $\mathfrak{B}^*$, as before, using inequality (3.6), we have the following estimates

$$\left|\frac{3}{2}s\lambda\int_0^T (p(p\beta_x\xi)_x)_x\big|_{x=L}|w_x(t,L)|^2dt\right| \lesssim s\lambda^3\int_0^T p^2(L)\beta_x^3(L)\xi^3(t,L)|w_x(t,L)|^2dt.$$

For the term with mixed derivative, using inequality (3.6) and Young's inequality we get

$$\begin{aligned}\left|3s\lambda\int_0^T (p^2(\beta_x\xi)_x)\big|_{x=L}w_x(t,L)w_{xx}(t,L)dt\right| &\lesssim s^2\lambda^3\int_0^T p^2(L)\beta_x^3(L)\xi^3(t,L)|w_x(t,L)|^2dt \\ &+\lambda\int_0^T p^2(L)\beta_x(L)\xi(t,L)|w_{xx}(t,L)|^2dt.\end{aligned}$$

The analogous bounds hold for the terms evaluated at $x=0$. We readily get for $s\geq s_0$,

$$|\mathfrak{B}^*| \lesssim \frac{1}{s}\mathfrak{B}_0+\frac{1}{s}\mathfrak{B}_L.$$

Then taking s_0 large enough, from (3.9), for any $s\geq s_0$ and $\lambda\geq\lambda_0$ it holds

$$\begin{aligned}\|w\|^2_{s,\lambda,\xi}+\mathfrak{B}_0+\mathfrak{B}_L+\mathfrak{B}_\Gamma &\lesssim \langle \mathcal{L}_1 w, \mathcal{L}_2 w\rangle_{L^2(Q)} \\ &+\iint_{(0,T)\times\omega_0}(s^5\lambda^6\xi^5|w|^2dxdt+s^3\lambda^4\xi^3|w_x|^2dxdt+s\lambda^2\xi|w_{xx}|^2)dxdt. \quad (3.10)\end{aligned}$$

3.1.5. *Treatment of the terms at the interface.* We introduce the following notation for the weighted norm at the interface

$$|w|^2_{\Gamma,s,\lambda,\xi} = \sum_{a\in\Gamma}\int_0^T\left(s^5\lambda^5\xi^5(t,a)|w(t,a)|^2+s^3\lambda^3\xi^3(t,a)|w_x(t,a^-)|^2+s\lambda\xi(t,a)|w_{xx}(t,a^-)|^2\right)dt.$$

First of all, notice that since both β and w satisfy the transmission conditions, we have

$$\int_0^T[p\beta_x\xi w_t w_x]_a dt = 0,\ \ \forall a\in\Gamma.$$

Now, the transmission conditions, allow us to rewrite the remaining terms of $\mathfrak{B}_\Gamma^{dom}$ as follows

$$\frac{3}{2}s^5\lambda^5\int_0^T\xi^5(t,a)[p^2\beta_x^5]_a|w(t,a)|^2dt = \frac{3}{2}s^5\lambda^5\int_0^T p_+^2\beta_x^4(a^+)[\beta_x]_a\xi^5(t,a)|w(t,a)|^2dt,$$

$$4s^3\lambda^3\int_0^T \xi^3(t,a)[p^2\beta_x^3|w_x|^2]_a dt = 4s^3\lambda^3\int_0^T p_+\beta_x^2(a^+)[\beta_x]_a\xi^3(t,a)p_-|w_x(t,a^-)|^2 dt,$$

$$\frac{3}{2}s\lambda\int_0^T \xi(t,a)[p^2\beta_x|w_{xx}|^2]_a dt = \frac{3}{2}s\lambda\int_0^T [\beta_x]_a\xi(t,a)p_-^2|w_{xx}(t,a^-)|^2 dt,$$

$$s^3\lambda^3\int_0^T [p^2\beta_x^3 w_{xx}]_a\xi^3(t,a)w(t,a)dt = s^3\lambda^3\int_0^T p_+\beta_x^2(a^+)[\beta_x]_a\xi^3(t,a)p_-w_{xx}(a^-)w(t,a)dt.$$

Let us define the vector function

$$\vec{\boldsymbol{w}}(t,a) = (s^2\lambda^2\xi^2(t,a)w(t,a), s\lambda\xi(t,a)\sqrt{p_-}w_x(t,a^-), p_-w_{xx}(t,a^-))^{\mathrm{tr}},$$

for $(t,a)\in(0,T)\times\Gamma$. By the above computations we can write

$$\mathfrak{B}_\Gamma^{dom}(a) = \frac{3}{2}s\lambda\int_0^T \xi(t,a)\big(\boldsymbol{A}\vec{\boldsymbol{w}}(t,a),\vec{\boldsymbol{w}}(t,a)\big)_{\mathbb{R}^3}dt,$$

where $\boldsymbol{A}$ is defined by

$$\boldsymbol{A}(a) := \begin{pmatrix} [p^2\beta_x^5]_a & 0 & \frac{1}{3}[p\beta_x^3]_a \\ 0 & \frac{8}{3}[p\beta_x^3]_a & 0 \\ \frac{1}{3}[p\beta_x^3]_a & 0 & [\beta_x] \end{pmatrix}. \tag{3.11}$$

No matter where ω_0 is located, since β satisfies the transmission conditions and Hypothesis $\mathfrak{M}$ is enforced, we have $[\beta_x]_a > 0$ for any $a\in\Gamma$. From this observation, given that $\boldsymbol{A}(a)$ is a symmetric matrix, by using Sylvester's criterion, it is not difficult to arrive that $\boldsymbol{A}(a)$ is a positive definite matrix. We thus choose $\gamma>0$ to be the minimum over the lower bounds of the associated quadratic form to $\boldsymbol{A}(a)$. We thus obtain, uniformly in $a\in\Gamma$, that

$$s\lambda\int_0^T \xi(t,a)\big(\boldsymbol{A}\vec{\boldsymbol{w}}(t,a),\vec{\boldsymbol{w}}(t,a)\big)_{\mathbb{R}^3}dt \geq \gamma s\lambda\int_0^T \xi(t,a)|\vec{\boldsymbol{w}}(t,a)|_{\mathbb{R}^3}^2 dt.$$

As we did with the boundary terms, by using inequality (3.6) and Young's inequality, we get

$$|\mathfrak{B}_\Gamma^{low}(a)| \lesssim \left(\frac{1}{s\lambda^2}+\frac{1}{s^2\lambda}+\frac{1}{s^2}\right)s^5\lambda^5\int_0^T \xi^5(t,a)|w(t,a)|^2 dt$$
$$+\frac{1}{s^2\lambda}s^3\lambda^3\int_0^T \xi^3(t,a)|w_x(t,a^-)|^2 dt + \frac{1}{\lambda}s\lambda\int_0^T \xi(t,a)|w_{xx}(t,a^-)|^2 dt$$

Since the above estimates are uniform with respect to $a\in\Gamma$, by choosing s_0 and λ_0 large enough yields that, for any $s\geq s_0$ and $\lambda\geq\lambda_0$,

$$|w|_{\Gamma,s,\lambda,\xi}^2 \lesssim \mathfrak{B}_\Gamma.$$

3.1.6. *Back to the original variable.* Gathering the inequalities obtained in the previous steps, we proved the following estimate for the conjugated operator.

Proposition 3.3. *Let (ω,p) satisfy Hypothesis $\mathfrak{M}$ and let $\omega_0 \Subset \omega$. There exist $s_0>0$, $\lambda_0>0$ and a constant $C>0$ depending on L, T, p, s_0, λ_0 and $\|\beta\|_{C^3([0,L]\setminus\Gamma)}$, such that for all $w\in\mathcal{V}_s$ we have*

$$C\big(\|\mathcal{L}_1 w\|_{L^2(Q)}^2 + \|\mathcal{L}_2 w\|_{L^2(Q)}^2 + \|w\|_{s,\lambda,\xi}^2 + |w|_{\Gamma,s,\lambda,\xi}^2\big)$$
$$\leq \|\mathcal{L}_\eta w\|_{L^2(Q)}^2 + \iint_{(0,T)\times\omega}(s^5\lambda^6\xi^5|w|^2 dxdt + s^3\lambda^4\xi^3|w_x|^2 dxdt + s\lambda^2\xi|w_{xx}|^2)dxdt,$$

for any $s\geq s_0$ and $\lambda\geq\lambda_0$.

Now we go back to the original variable. Recall that $u = e^{s\eta}w$ belongs to $\mathcal{V}$ and $\mathcal{L}_\eta w = e^{-s\eta}\mathcal{L}u$. Straightforward computations lead us to

$$e^{-2s\eta}|u_x|^2 \lesssim |w_x|^2 + s^2\lambda^2\xi^2|w|^2,$$
$$e^{-2s\eta}|u_{xx}|^2 \lesssim |w_{xx}|^2 + s^2\lambda^2\xi^2|w_x|^2 + s^4\lambda^4\xi^2|w|^2,$$

for all $(t,x) \in Q'$, from which we get

$$\iint_Q e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2 + s^3\lambda^4\xi^3|u_x|^2 + s\lambda^2\xi|u_{xx}|^2)dxdt \lesssim \iint_Q (s^5\lambda^6\xi^5|w|^2 + s^3\lambda^4\xi^3|w_x|^2 + s\lambda^2\xi|w_{xx}|^2)dxdt.$$

Similarly,

$$e^{2s\eta}|w_x|^2 \lesssim |u_x|^2 + s^2\lambda^2\xi^2|u|^2,$$
$$e^{2s\eta}|w_{xx}|^2 \lesssim |u_{xx}|^2 + s^2\lambda^2\xi^2|u_x|^2 + s^4\lambda^4\xi^2|u|^2,$$

for all $(t,x) \in Q'$. Thus

$$\iint_{(0,T)\times\omega} (s^5\lambda^6\xi^5|w|^2 + s^3\lambda^4\xi^3|w_x|^2 + s\lambda^2\xi|w_{xx}|^2)dxdt \lesssim \iint_{(0,T)\times\omega} e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2 + s^3\lambda^4\xi^3|u_x|^2 + s\lambda^2\xi|u_{xx}|^2)dxdt.$$

From the above estimates, Proposition 3.3 directly implies Theorem 1.1.

4. Control to the trajectories

The aim of this section is to prove the controllability result Theorem 1.2. To this end we will consider the following two relevant systems. The first one corresponds to the linearized system to (1.7) around the aimed trajectory $\overline{y}$, which is

$$\begin{cases} z_t + p(x)z_{xxx} + z_x + (\overline{y}z)_x = h + \mathbb{1}_\omega v, & (t,x) \in (0,T)\times(0,L), \\ z(t,0) = z(t,L) = z_x(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{4.1}$$

coupled by (TC), where $v \in L^2(0,T;L^2(0,L))$ is the control and h is a source in some appropriate weighted space. The second relevant system corresponds to the adjoint system associated to (4.1)

$$\begin{cases} -\varphi_t - p(x)\varphi_{xxx} - \varphi_x - \overline{y}\varphi_x = g, & (t,x) \in (0,T)\times(0,L), \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_T(x), & x \in (0,L). \end{cases} \tag{4.2}$$

coupled by (TC), with appropriate initial data φ_T and source term g. The strategy follows a classical duality argument which is briefly described below:

1. (1) We establish a suitable Carleman estimate for the adjoint system (4.2).
2. (2) By means of the Carleman estimate, we obtain an observability inequality for (4.2). We then employ a variational approach to establish the null controllability of the linearized system (4.1) with a right-hand side decaying near $t = T$.
3. (3) We then apply a local inversion result in a suitable functional setting - inherited from the variational approach - to obtain the null controllability of the nonlinear system (1.7).

In this section, we closely follow Cerpa, Montoya and Zhang [CMZ20] and show their arguments can be adapted to the piecewise constant case. The main point being the regularity estimates provided by Proposition 2.12, which combined with the Carleman estimate of Theorem 1.1 will allow us to obtain a suitable one-parameter Carleman estimate and henceforth carry out the strategy.

4.1. **A suitable observability inequality.** Let $\overline{y} \in \mathcal{X}^3_{\Gamma,T}(0,L)$, whose existence is guaranteed by Proposition 2.6. Let us introduce the operator $\mathcal{L} : \mathcal{V} \to L^2(Q)$ given by

$$\mathcal{L}z = z_t + p(x)z_{xxx} + z_x + (\overline{y}z)_x, \tag{4.3}$$

defined on the space of functions

$$\mathcal{V} = \{z \in L^2(0,T; H^3_\Gamma(0,L)) \mid \mathcal{L}z \in L^2(Q),\ z(0) = z(L) = z'(L) = 0 \text{ and } z \text{ satisfies (TC)}\}.$$

In what follows, let $\omega_0 \Subset \omega$ be non-empty and open, $\kappa \in (1,2)$ and β be constructed by Lemma 3.1. From now on, let us fix $\lambda \geq \lambda_0$ large enough so the Carleman estimate of Theorem 1.1 holds true with the weights η and ξ introduced in (1.5). Let us denote

$$\widehat{\eta}(t) = \max_{x\in[0,L]} \eta(t,x), \quad \check{\eta}(t) = \min_{x\in[0,L]} \eta(t,x), \quad \zeta(t) = \frac{1}{t^2(T-t)^2}, \tag{4.4}$$

We have the following one-parameter Carleman estimate.

Proposition 4.1. *Let (ω, p) satisfy Hypothesis $\mathfrak{M}$. Let $\overline{y} \in \mathcal{X}^3_{\Gamma,T}(0,L)$ be a solution of (1.8). There exist $s_0 > 0$ and $C > 0$ depending on ω, Γ, L, T, p, s_0, λ_0 and $\|\beta\|_{C^3_\Gamma([0,L]\setminus\Gamma)}$ such that for any $\varphi_T \in \mathcal{D}(\mathcal{A}^*)$ and $g \in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$, the corresponding solution φ to (4.2) satisfies*

$$\begin{aligned} &\iint_Q e^{-4s\widehat{\eta}}(s^5\zeta^5|\varphi|^2 + s^3\zeta^3|\varphi_x|^2 + s\zeta|\varphi_{xx}|^2)dxdt \\ &\qquad \leq C\left(\iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + s^7\iint_{(0,T)\times\omega} e^{-6s\check{\eta}+2s\widehat{\eta}}\zeta^7|\varphi|^2dxdt\right), \end{aligned} \tag{4.5}$$

for any $s \geq s_0$.

Proof. The proof is made in two steps: we first decompose the solution of (4.2) to have a regular source term with suitable decay in t and for which we will apply the Carleman estimate. This decomposition will then allow us to employ a bootstrap argument to estimate the local terms coming from the higher order norms on the right-hand side of the Carleman estimate.

Step 1. Decomposition of the solution. Let us decompose the solution φ of (4.2), with the aim of obtaining L^2 regularity on the right-hand side of (4.2). Let us introduce z and u, solutions of

$$\begin{cases} -z_t - p(x)z_{xxx} - z_x - \overline{y}z_x = \rho_0 g, & (t,x) \in (0,T)\times(0,L), \\ z(t,0) = z(t,L) = z_x(t,0) = 0, & t \in (0,T), \\ z(T,x) = 0, & x \in (0,L), \end{cases} \tag{4.6}$$

and

$$\begin{cases} -u_t - p(x)u_{xxx} - u_x - \overline{y}u_x = (-\rho_0)_t\varphi, & (t,x) \in (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,0) = 0, & t \in (0,T), \\ u(T,x) = 0, & x \in (0,L), \end{cases} \tag{4.7}$$

both of them coupled by the corresponding transmission conditions (TC), with $\rho_0(t) := e^{-s\widehat{\eta}}$. By uniqueness, we have $\rho_0\varphi = u + z$. For the first system, using the regularity result for (4.6), we have

$$\|z\|^2_{L^2(0,T;H^2_\Gamma(0,L))} \leq C\|\rho_0 g\|^2_{L^2(Q)}. \tag{4.8}$$

Now, we apply the Carleman estimate of Theorem 1.1 for solutions of (4.7), with the new weights (4.4) and fixed λ as above, obtaining

$$C\iint_Q e^{-2s\widehat{\eta}}(s^5\zeta^5|u|^2+s^3\zeta^3|u_x|^2+s\zeta|u_{xx}|^2)dxdt \leq \iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + \iint_{(0,T)\times\omega} e^{-2s\check{\eta}}(s^5\zeta^5|u|^2dxdt+s^3\zeta^3|u_x|^2dxdt+s\zeta|u_{xx}|^2)dxdt. \quad (4.9)$$

Observe that on the right-hand side, we used $|(\rho_0)_t\varphi|\leq Cs\xi^{3/2}|\rho_0\varphi|$ followed by the relation $\rho_0\varphi = u+z$, to then absorb the term containing u for s large enough and conclude using estimate (4.8).

Step 2. Local estimates. By classical interpolation (see Lions-Magenes [LM72, Section 9])

$$H^1(\omega)=(H^3(\omega),L^2(\omega))_{2/3,2},\ H^2(\omega)=(H^3(\omega),L^2(\omega))_{1/3,2}.$$

Let $\varepsilon>0$. By Young's inequality with $(p,q)=(3/2,3)$ we get

$$s^3\iint_{(0,T)\times\omega}\zeta^3e^{-2s\check{\eta}}|u_x|^2dxdt \leq s^3\int_0^T\zeta^3e^{-2s\check{\eta}}\|u\|^{4/3}_{L^2(\omega)}\|u\|^{2/3}_{H^3(\omega)}dt \leq C_\varepsilon s^{11/2}\int_0^T\zeta^{11/2}e^{-3s\check{\eta}+s\widehat{\eta}}\|u\|^2_{L^2(\omega)}dt+\varepsilon s^{-2}\int_0^T\zeta^{-2}e^{-s2\widehat{\eta}}\|u\|^2_{H^3(\omega)}dt.$$

Similarly, with $(p,q)=(3,3/2)$ we get

$$s\iint_{(0,T)\times\omega}e^{-2s\check{\eta}}\zeta|u_{xx}|^2dxdt \leq s\int_0^T\zeta e^{-2s\check{\eta}}\|u\|^{2/3}_{L^2(\omega)}\|u\|^{4/3}_{H^3(\omega)}dt \leq C_\varepsilon s^7\int_0^T\zeta^7e^{-6s\check{\eta}+4s\widehat{\eta}}\|u\|^2_{L^2(\omega)}dt+\varepsilon s^{-2}\int_0^T\zeta^{-2}e^{-2s\widehat{\eta}}\|u\|^2_{H^3(\omega)}dt.$$

From the Carleman estimate and the inequalities above, we get

$$C\iint_Q e^{-2s\eta}(s^5\zeta^5|u|^2+s^3\zeta^3|u_x|^2+s\zeta|u_{xx}|^2)dxdt \leq \iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt+s^7\iint_{(0,T)\times\omega}\zeta^7e^{-6s\check{\eta}+4s\widehat{\eta}}|u|^2dxdt+\varepsilon\left(s^{-2}\int_0^T\zeta^{-2}e^{-2s\widehat{\eta}}\|u\|^2_{H^3(\omega)}dt\right).$$

We want to estimate the local term containing $\|u\|^2_{H^3(\omega)}$. By looking at the weights accompanying the local H^3-norm, let us introduce $\widehat{u}=\widehat{\rho}(t)u$ with $\widehat{\rho}(t):=s^{-1/2}\lambda^{-1}\zeta^{-1/2}e^{-s\widehat{\eta}}$. We thus see that $\widehat{u}$ solves (4.11) with $\tilde{\rho}$ replaced by $\widehat{\rho}$,

$$\begin{cases} -\widehat{u}_t-p(x)\widehat{u}_{xxx}-\widehat{u}_x-\overline{y}\widehat{u}_x=\widehat{\rho}(-\rho_0)_t\varphi-\widehat{\rho}_tu, & (t,x)\in(0,T)\times(0,L),\\ \widehat{u}(t,0)=\widehat{u}(t,L)=\widehat{u}_x(t,0)=0, & t\in(0,T),\\ \widehat{u}(T,x)=0, & x\in(0,L), \end{cases} \quad (4.10)$$

coupled by the corresponding transmission conditions (TC). Since $\varphi\in C([0,T],\mathcal{D}(\mathcal{A}^*))$, using the regularity estimates given by Proposition 2.12, we have

$$\|s^{-1/2}\zeta^{-1/2}e^{-s\widehat{\eta}}u\|^2_{L^2(0,T;H^3_\Gamma(0,L))}=\|\widehat{u}\|^2_{L^2(0,T;H^3_\Gamma(0,L))} \leq C\big(\|s^{1/2}\zeta e^{-s\widehat{\eta}}u\|^2_{L^2(0,T;H^1(0,L))}+\|s^{1/2}\zeta e^{-2s\widehat{\eta}}\varphi\|^2_{L^2(0,T;H^1(0,L))}\big).$$

We are then led to define $\tilde{u} = \tilde{\rho}(t)u$ with $\tilde{\rho}(t) = s^{1/2}\zeta e^{-s\widehat{\eta}}$ aiming to estimate the first term of the right-hand side in the inequality above. We see that $\tilde{u}$ is the solution of

$$\begin{cases} -\tilde{u}_t - p(x)\tilde{u}_{xxx} - \tilde{u}_x - \overline{y}\tilde{u}_x = \tilde{\rho}(-\rho_0)_t\varphi - \tilde{\rho}_t u, & (t,x) \in (0,T)\times(0,L), \\ \tilde{u}(t,0) = \tilde{u}(t,L) = \tilde{u}_x(t,0) = 0, & t \in (0,T), \\ \tilde{u}(T,x) = 0, & x \in (0,L), \end{cases} \tag{4.11}$$

coupled by the corresponding transmission conditions (TC). As $|(\rho_0)_t| \lesssim s\zeta^{3/2}e^{-s\widehat{\eta}}$, we get

$$|\tilde{\rho}_t| = s^{1/2}|\zeta_t\rho_0(t) + \zeta(\rho_0)_t| \lesssim \big(s^{1/2}\zeta^{3/2} + s^{3/2}\zeta^{5/2}\big)\rho_0 \lesssim s^{3/2}\zeta^{5/2}e^{-s\widehat{\eta}}.$$

Using once again Proposition 2.12, we obtain

$$\|s^{1/2}\zeta e^{-s\widehat{\eta}}u\|^2_{L^2(0,T;H^2_\Gamma(0,L))} = \|\tilde{u}\|^2_{L^2(0,T;H^2_\Gamma(0,L))} \le C\big(\|s^{3/2}\zeta^{5/2}e^{-s\widehat{\eta}}u\|^2_{L^2(Q)} + \|s^{3/2}\zeta^{5/2}e^{-2s\widehat{\eta}}\varphi\|^2_{L^2(Q)}\big).$$

Gathering the above inequalities, we have

$$\|s^{-1/2}\zeta^{-1/2}e^{-s\widehat{\eta}}u\|^2_{L^2(0,T;H^3_\Gamma(0,L))} \le C\big(\|s^{3/2}\zeta^{5/2}e^{-s\widehat{\eta}}u\|^2_{L^2(Q)} + \|s^{3/2}\zeta^{5/2}e^{-2s\widehat{\eta}}\varphi\|^2_{L^2(Q)} + \|s^{1/2}\zeta e^{-2s\widehat{\eta}}\varphi\|^2_{L^2(0,T;H^1(0,L))}\big).$$

Since $(s,t) \mapsto s^{3/2}\zeta^{5/2}e^{-s\widehat{\eta}}$ is bounded, using (4.8) we see that the right-hand side of the inequality above is bounded by the left-hand side of the Carleman estimate (4.9) and $\|\rho_0 g\|^2_{L^2(Q)}$. Therefore

$$\iint_Q e^{-2s\eta}(s^5\zeta^5|u|^2 + s^3\zeta^3|u_x|^2 + s\zeta|u_{xx}|^2)dxdt + \|s^{-1/2}\zeta^{-1/2}e^{-s\widehat{\eta}}u\|^2_{L^2(0,T;H^3_\Gamma(0,L))}$$
$$\le C\left(\iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + s^7\iint_{(0,T)\times\omega} e^{-6s\check{\eta}+4s\widehat{\eta}}\zeta^7|u|^2dxdt\right) + \varepsilon\left(s^{-2}\int_0^T \zeta^{-2}e^{-2s\widehat{\eta}}\|u\|^2_{H^3(\omega)}dt\right).$$

Choosing $\varepsilon > 0$ small enough, the last term on the right-hand side above, can be absorbed by the last term on the left-hand side above. To return to the φ variable, we use $\rho_0\varphi = z + u$ and estimate (4.8) to get

$$\iint_Q e^{-4s\widehat{\eta}}(s^5\xi^5|\varphi|^2 + s^3\xi^3|\varphi_x|^2 + s\xi|\varphi_{xx}|^2)dxdt$$
$$\le C\left(\iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + \iint_Q e^{-2s\eta}(s^5\zeta^5|u|^2 + s^3\zeta^3|u_x|^2 + s\zeta|u_{xx}|^2)dxdt\right).$$

Once again, using estimate (4.8) and that $(s,t) \mapsto s^7 e^{-6s\check{\eta}+4s\widehat{\eta}}\zeta^7$ is bounded for $s \ge s_0$ and $t \in (0,T)$, we obtain

$$s^7\iint_{(0,T)\times\omega} e^{-6s\check{\eta}+4s\widehat{\eta}}\zeta^7|u|^2dxdt \le C\left(\iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + s^7\iint_{(0,T)\times\omega} e^{-6s\check{\eta}+2s\widehat{\eta}}\zeta^7|\varphi|^2dxdt\right).$$

Putting together the three last estimates, we arrive to inequality (4.5), finishing the proof. □

For notational convenience, we introduce $\mathcal{L}^* : \mathcal{V}^* \to L^2(Q)$, the adjoint operator to (4.3)

$$\mathcal{L}^*\psi := -\psi_t - p(x)\psi_{xxx} - \psi_x - \overline{y}\psi_x, \tag{4.12}$$

defined on the space of functions

$$\mathcal{V}^* := \{\psi \in L^2(0,T;H^3_\Gamma(0,L)) \mid \mathcal{L}^*\psi \in L^2(Q),\ \psi(0) = \psi(L) = \psi'(0) = 0 \text{ and } \psi \text{ satisfies (TC)}\}.$$

We now introduce weight that does not vanish at $t = 0$. Let $\ell \in C^1([0,T])$ be a positive function in $[0,T)$ defined by

$$\ell(t) = \begin{cases} T^2/4 & t \in [0,T/2] \\ t(T-t) & t \in [T/2,T] \end{cases}$$

We then consider

$$\alpha(t,x) = (e^{\kappa\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)})\tau(t), \quad \tau(t) = \frac{1}{\ell(t)}, \quad \widehat{\alpha}(t) = \max_{x\in[0,L]} \alpha(t,x), \quad \breve{\alpha}(t) = \min_{x\in[0,L]} \alpha(t,x).$$

We further ask that $\lambda \geq \kappa^2/\|\beta\|_\infty$, where κ is the parameter used in Lemma 3.1. Thus, from now on we assume that $\lambda \geq \max\{\lambda_0, \kappa^2/\|\beta\|_\infty\}$ and therefore $2\widehat{\alpha} < 3\breve{\alpha}$ holds. As a consequence of Proposition 4.1 we have the following weighted observability inequality.

Lemma 4.2. *Under the assumptions of Lemma 4.2, there exist s and C such that every solution φ of (4.2) satisfies*

$$\iint_Q e^{-4s\widehat{\alpha}}(\tau^5|\varphi|^2 + \tau^3|\varphi_x|^2 + \tau|\varphi_{xx}|^2)dxdt + \|\varphi(0)\|^2_{L^2(0,L)}$$
$$\leq C\left(\iint_Q e^{-2s\widehat{\alpha}}|g|^2dxdt + \iint_{(0,T)\times\omega} e^{-6s\breve{\alpha}+2s\widehat{\alpha}}\tau^7|\varphi|^2dxdt\right). \quad (4.13)$$

Proof. By construction, $\eta = \alpha$ and $\tau = \zeta$ in $(T/2,T)\times[0,L]$. Therefore, as a consequence of Proposition 4.1 we get

$$\int_{T/2}^T\int_0^L e^{-4s\widehat{\alpha}}(s^5\tau^5|\varphi|^2 + s^3\tau^3|\varphi_x|^2 + s\tau|\varphi_{xx}|^2)dxdt$$
$$= \int_{T/2}^T\int_0^L e^{-4s\widehat{\eta}}(s^5\zeta^5|\varphi|^2 + s^3\zeta^3|\varphi_x|^2 + s\zeta|\varphi_{xx}|^2)dxdt$$
$$\leq C\left(\iint_Q e^{-2s\widehat{\eta}}|g|^2dxdt + s^7\iint_{(0,T)\times\omega} e^{-6s\breve{\eta}+4s\widehat{\eta}}\zeta^7|\varphi|^2dxdt\right).$$

From now on, let us fix $s \geq s_0$. By construction of the weights, we only need to focus the analysis on $(0,T/2)$. Using inequalities $e^{-2s\widehat{\eta}} \leq C$ and $e^{-6s\breve{\eta}+4s\widehat{\eta}}\zeta^7 \geq C$ in $[0,T/2]$, followed by the fact that τ is constant in $[0,T/2]$, we get

$$\int_{T/2}^T\int_0^L e^{-4s\widehat{\alpha}}(s^5\tau^5|\varphi|^2 + s^3\tau^3|\varphi_x|^2 + s\tau|\varphi_{xx}|^2)dxdt$$
$$\leq C\left(\iint_Q e^{-2s\widehat{\alpha}}|g|^2dxdt + \iint_{(0,T)\times\omega} e^{-6s\breve{\alpha}+4s\widehat{\alpha}}\tau^7|\varphi|^2dxdt\right). \quad (4.14)$$

Let us take a cutoff $\chi \in C^1([0,T])$ such that $\chi \equiv 1$ in $[0,T/2]$ and $\chi \equiv 0$ in $[3T/4,T]$. Observe that $\chi\varphi \in \mathcal{V}^*$, $\chi(T)\varphi(T,\cdot) = 0$ and $\mathcal{L}^*(\chi\varphi) = \chi\mathcal{L}^*\varphi - \chi'\varphi$. Thus, given that $g \in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$, by semigroup estimates we get

$$\|\chi\varphi\|_{C([0,T],L^2(0,L))} \leq C\|\chi g - \chi'\varphi\|_{L^2(0,T;L^2(0,L))},$$

from which follows

$$\|\varphi\|_{C([0,T/2],L^2(0,L))} \leq C\|g\|_{L^2(0,3T/4;L^2(0,L))} + \|\varphi\|_{L^2(T/2,3T/4;L^2(0,L))}.$$

By employing Proposition 2.12 and the above estimate, we obtain

$$\|\varphi(0)\|^2_{L^2(0,L)} + \|\varphi\|^2_{L^2(0,T/2;H^2_\Gamma(0,L))} \leq C\big(\|g\|^2_{L^2(0,3T/4;L^2(0,L))} + \|\varphi\|^2_{L^2(T/2,3T/4;L^2((0,L))}\big).$$

Taking into account that

$$\tau^5 e^{-2s\widehat{\alpha}} \geq C > 0, \ \forall t \in [T/2,3T/4] \ \text{ and } \ e^{-4s\widehat{\alpha}} \geq C > 0, \ \forall t \in [0,3T/4],$$

we arrive to

$$C\int_0^{T/2}\int_0^L e^{-4s\widehat{\alpha}}(s^5\tau^5|\varphi|^2+s^3\tau^3|\varphi_x|^2+s\tau|\varphi_{xx}|^2)dxdt+\|\varphi(0)\|^2_{L^2(0,L)}$$
$$\le \int_0^{3T/4}\int_0^L e^{-2s\widehat{\alpha}}|g|^2dxdt+\int_{T/2}^{3T/4}\int_0^L e^{-6s\check{\alpha}+4s\widehat{\alpha}}\tau^5|\varphi|^2dxdt. \quad (4.15)$$

Inequality (4.13) then follows, upon adjusting $s\ge s_0$ if necessary, by combining (4.14) and (4.15). □

4.2. **Null controllability of the linearized system.** Let us introduce the space

$$\mathcal{E}:=\{(z,v)\mid e^{s\widehat{\alpha}}z\in L^2(Q),\ \tau^{-9/2}e^{3s\check{\alpha}-s\widehat{\alpha}}v\mathbb{1}_\omega\in L^2(Q),$$
$$e^{s\widehat{\alpha}}\tau^{-3/2}z\in\mathcal{X}^0_T(0,L),\ e^{2s\widehat{\alpha}}\tau^{-5/2}(\mathcal{L}z-\mathbb{1}_\omega v)\in L^2(0,T;H^{-1}(0,L))\},$$

which is a Banach space when equipped with the norm whose square is given by

$$\|(z,v)\|^2_{\mathcal{E}}=\|e^{s\widehat{\alpha}}z\|^2_{L^2(Q)}+\|\tau^{-9/2}e^{3s\check{\alpha}-s\widehat{\alpha}}v\mathbb{1}_\omega\|^2_{L^2(Q)}$$
$$+\|e^{s\widehat{\alpha}}\tau^{-3/2}z\|_{\mathcal{X}^0_T(0,L)}+\|e^{2s\widehat{\alpha}}\tau^{-5/2}(\mathcal{L}z-v\mathbb{1}_\omega)\|^2_{L^2(0,T;H^{-1}(0,L))}.$$

We now aim to solve (4.1) in the space $\mathcal{E}$ with a right-hand side in an appropriate weighted space. Indeed, in such case, from which the null controllability of the system follows given that $e^{s\widehat{\alpha}}\tau^{-3/2}z\in C([0,T],L^2(0,L))$ implies that $z(T,\cdot)=0$.

Proposition 4.3. *Let (ω,p) satisfy Hypothesis $\mathfrak{M}$ and let $T>0$. For any $z_0\in L^2(0,L)$ and $e^{2s\widehat{\alpha}}\tau^{-5/2}h\in L^2(Q)$, there exists a function $v\in L^2(0,T;L^2(\omega))$ such that the associated solution (z,v) to (4.1) satisfies $(z,v)\in\mathcal{E}$. Furthermore, there exists $C>0$ such that*

$$\|v\|_{L^2(0,T;L^2(\omega))}\le C\big(\|z_0\|_{L^2(0,L)}+\|h\|_{L^2(Q)}\big). \quad (4.16)$$

Proof. Set $\mathcal{Q}_0$ to be the space of functions $\varphi\in C^3([0,T]\times([0,L]\setminus\Gamma))$ such that:

- $\varphi_{|_{I_k}}\in C^3([0,T]\times\overline{I}_k)$, $k\in[\![0,N-1]\!]$;
- φ satisfies the transmission conditions (TC);
- φ satisfies the boundary conditions $\varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0$, $t\in(0,T)$.

Let us introduce the bilinear form $\boldsymbol{a}(\cdot,\cdot)$ on $\mathcal{Q}_0$

$$\boldsymbol{a}(\hat{\varphi},w):=\iint_Q e^{-2s\widehat{\alpha}}(\mathcal{L}^*\hat{\varphi})(\mathcal{L}^*w)dxdt+\iint_{\omega\times(0,T)}e^{-6s\check{\alpha}+2s\widehat{\alpha}}\tau^7\hat{\varphi}wdxdt,\quad \forall(\hat{\varphi},w)\in\mathcal{Q}_0\times\mathcal{Q}_0,$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ defined in (4.3). Observe that Carleman inequality (4.13) is applicable for any $w\in\mathcal{Q}_0$, thus

$$\iint_Q\tau^5e^{-4s\widehat{\alpha}}|w|^2dxdt+\|w(0)\|^2_{L^2(0,L)}\le C\boldsymbol{a}(w,w),\ \forall w\in\mathcal{Q}_0. \quad (4.17)$$

In particular, a unique continuation property holds, in other words, $\boldsymbol{a}(w,w)=0$ implies that $w=0$ in $\mathcal{Q}_0$. Further, observe that the bound given here above implies that $\boldsymbol{a}(\cdot,\cdot):\mathcal{Q}_0\times\mathcal{Q}_0\to\mathbb{R}$ is a coercive bilinear form. It being symmetric as well, $\boldsymbol{a}(\cdot,\cdot)$ defines an inner product in $\mathcal{Q}_0$. We introduce $\mathcal{Q}$ as the completion of $\mathcal{Q}_0$ for the form induced by $\boldsymbol{a}(\cdot,\cdot)$, which we denote by $\|\cdot\|_{\mathcal{Q}}$. Certainly, $\mathcal{Q}$ is a Hilbert space and $\boldsymbol{a}(\cdot,\cdot)$ is a continuous and coercive bilinear form on $\mathcal{Q}$.

Let us introduce the linear form $\mathcal{G}$, given by

$$\langle\mathcal{G},w\rangle:=\iint_Q hwdxdt+\int_0^L z_0(x)w(0,x)dx,\ \forall w\in\mathcal{Q}.$$

Given that $e^{2s\widehat{\alpha}}\tau^{-5/2}h \in L^2(Q)$, by the Carleman inequality (4.13), the linear form $w \in \mathcal{Q} \mapsto \langle G, w\rangle \in \mathbb{R}$ is well-defined and continuous. Indeed,

$$|\langle \mathcal{G}, w\rangle| \leq \|e^{s\widehat{\alpha}}\tau^{-5/4}h\|_{L^2(Q)}\|e^{-s\widehat{\alpha}}\tau^{5/4}w\|_{L^2(Q)} + \|z_0\|_{L^2(0,L)}\|w(0,\cdot)\|_{L^2(0,L)},$$

and using inequality (4.17) along with the density of $\mathcal{Q}_0$ in $\mathcal{Q}$, we have

$$|\langle \mathcal{G}, w\rangle| \leq \left(\|e^{s\widehat{\alpha}}\tau^{-5/4}h\|_{L^2(Q)} + \|z_0\|_{L^2(0,L)}\right)\|w\|_{\mathcal{Q}}, \tag{4.18}$$

valid for any $w \in \mathcal{Q}$. Applying Lax-Milgram's lemma, there exists a unique $\hat{\varphi} \in \mathcal{Q}$ such that

$$\boldsymbol{a}(\hat{\varphi}, w) = \langle \mathcal{G}, w\rangle, \ \forall w \in \mathcal{Q}. \tag{4.19}$$

Introduce

$$\begin{cases} \hat{z} := e^{-2s\widehat{\alpha}}\mathcal{L}^*\hat{\varphi}, & \text{in } Q, \\ \hat{v} := -e^{-6s\breve{\alpha}+2s\widehat{\alpha}}\tau^7\hat{\varphi}, & \text{in } (0,T)\times\omega. \end{cases}$$

Given that $\hat{\varphi} \in \mathcal{Q}$, we notice that the pair $(\hat{z}, \hat{v})$ verifies

$$\boldsymbol{a}(\hat{\varphi}, \hat{\varphi}) = \iint_Q e^{2s\widehat{\alpha}}|\hat{z}|^2 dxdt + \iint_{(0,T)\times\omega} e^{6s\breve{\alpha}-2s\widehat{\alpha}}\tau^{-7}|\hat{v}|^2 dxdt < +\infty. \tag{4.20}$$

Furthermore, we see that $\hat{z}$ is the unique solution by transposition of (4.1) with v replaced by $\hat{v}$. Indeed, from (4.19) we readily get the variational identity: for every $g \in L^2(Q)$ we have

$$\iint_Q \hat{z} g dxdt = \iint_Q (h + \hat{v}) w dxdt + \int_0^L z_0(x) w(0,x) dx,$$

with $w \in \mathcal{X}_T^0(0,L)$ solution of the adjoint (4.2) with right-hand side g and $w(T,\cdot) = 0$, whose existence is guaranteed by Proposition 2.12.

As a last step, we verify that $(\hat{z}, \hat{v}) \in \mathcal{E}$. From (4.20) we readily get $e^{s\widehat{\alpha}}\hat{z} \in L^2(Q)$ and $e^{3s\breve{\alpha}-s\widehat{\alpha}}\tau^{-7/2}\hat{v} \in L^2(Q)$. Moreover, using the equation and that $e^{2s\widehat{\alpha}}\tau^{-5/2}h \in L^2(Q)$, we readily get

$$e^{2s\widehat{\alpha}}\tau^{-5/2}(\mathcal{L}\hat{z} - \mathbb{1}_\omega \hat{v}) \in L^2(Q).$$

To check that $e^{s\widehat{\alpha}}\tau^{-3/2}\hat{z} \in \mathcal{X}_T^0(0,L)$, we define

$$z^* := e^{s\widehat{\alpha}}\tau^{-3/2}\hat{z} \ \text{ and } \ h^* = e^{s\widehat{\alpha}}\tau^{-3/2}(h + \hat{v}).$$

Observe that z^* satisfies the system

$$\begin{cases} z^*_t + p(x)z^*_{xxx} + z^*_x + (\overline{y}z^*)_x = h^* + (e^{s\widehat{\alpha}}\tau^{-3/2})_t\hat{z}, & (t,x) \in (0,T)\times(0,L), \\ z^*(t,0) = z^*(t,L) = z^*_x(t,L) = 0, & t \in (0,T), \\ z^*(0,x) = e^{s\widehat{\alpha}(0)}\tau^{-3/2}(0)\hat{z}_0(x), & x \in (0,L), \end{cases}$$

coupled by the corresponding transmission conditions (TC). Since $e^{s\widehat{\alpha}}h \in L^2(Q)$ and $2\widehat{\alpha} < 3\breve{\alpha}$, we get $h^* \in L^2(Q)$ and $(e^{s\widehat{\alpha}}\tau^{-3/2})_t\hat{z} \in L^2(Q)$. For $\hat{z}_0 \in L^2(0,L)$, Proposition 2.3 along with an argument similar to the one used in Proposition 2.12 give us $z^* \in \mathcal{X}_T^0(0,L)$.

By considering $\hat{v}$ as before, the bilinear form $\boldsymbol{a}$ and identity (4.19), we obtain estimate (4.16). ☐

4.3. Control of the nonlinear system.

The last step relies on a local inversion result.

Theorem 4.4. [FI96, Chapter I, Section 4, Theorem 4.1] *Suppose that $\mathcal{B}_1$, $\mathcal{B}_2$ are Banach spaces and $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ is a continuously differentiable map. We assume that for $b_1^0 \in \mathcal{B}_1$, $b_2^0 \in \mathcal{B}_2$ the equality*

$$\mathcal{F}(b_1^0) = b_2^0$$

holds and $\mathcal{F}'(b_1^0) : \mathcal{B}_1 \to \mathcal{B}_2$ is a surjective. Then there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ which satisfies the condition $\|b_2^0 - b_2\|_{\mathcal{B}_2} < \delta$ there exists a solution $b_1 \in \mathcal{B}_1$ of the equation

$$\mathcal{F}(b_1) = b_2.$$

We now prove the main control result for the nonlinear system.

Proof of Theorem 1.2. Let us set

$$\mathcal{B}_1 := \mathcal{E} \quad \text{and} \quad \mathcal{B}_2 = L^2\big(e^{2s\widehat{\alpha}}\tau^{-5/2}(0,T); L^2(0,L)\big) \times L^2(0,L)$$

and the operator $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ defined by

$$\mathcal{F}(y,v) = \big(z_t + p(x)z_{xxx} + z_x + (\overline{y}z)_x + zz_x - \mathbb{1}_\omega v, z(0)\big).$$

We now prove that $\mathcal{F}$ is of class $C^1(\mathcal{B}_1,\mathcal{B}_2)$. Let us assume that $\overline{y} \in \mathcal{X}_T^0(0,L)$. By linearity, it only remains to prove that the bilinear operator

$$\big((z^1,v^1),(z^2,v^2)\big) \in \mathcal{E} \times \mathcal{E} \longmapsto \frac{1}{2}(z^1z^2)_x \in L^2\big(e^{2s\widehat{\alpha}}\tau^{-5/2}(0,T); L^2(0,L)\big)$$

is continuous. Observe that

$$e^{2s\widehat{\alpha}}\tau^{-5/2}z \in \mathcal{X}_T^0(0,L),$$

for any $(z,v) \in \mathcal{E}$. By Sobolev embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$, we have

$$\begin{aligned}\|e^{2s\widehat{\alpha}}\tau^{-5/2}(z^1z^2)_x\|_{L^2(Q)} \le C \int_0^T &\big(e^{2s\widehat{\alpha}}\tau^{-3}\|z^1\|^2_{L^\infty(0,L)}e^{2s\widehat{\alpha}}\tau^{-3}\|z^2\|_{H^1(0,L)} + \\ &+ e^{2s\widehat{\alpha}}\tau^{-3}\|z^2\|^2_{L^\infty(0,L)}e^{2s\widehat{\alpha}}\tau^{-3}\|z^1\|_{H^1(0,L)}\big)dt \\ \le C\|z^1\|_{\mathcal{B}_1}\|z^2\|_{\mathcal{B}_1}.\end{aligned}$$

We are in position to apply Theorem 4.4, with $b_1^0 = (0,0) \in \mathcal{B}_1$ and $b_2 = 0 \in \mathcal{B}_2$. The derivative $\mathcal{F}'(0,0) : \mathcal{B}_1 \to \mathcal{B}_2$ is given by

$$\mathcal{F}(0,0)(z,v) = \big(z_t + p(x)z_{xxx} + z_x + (\overline{y}z)_x - \mathbb{1}_\omega v, z(0)\big), \ \forall (z,v) \in \mathcal{B}_1.$$

Thus, there exists $\delta > 0$ such that, if $\|z(0)\|_{L^2(0,L)} \le \delta$, we can find a control v such that the associated solution z of the nonlinear system (1.9) satisfies $z(T,\cdot) = 0$ on $(0,L)$. This finishes the proof. $\square$

5. Lipschitz stability in retrieving an unknown potential

In this section we follow [BCCM14]. A key point in the latter work is that some symmetry assumptions on the coefficient to recover and on the initial data are made, in order to avoid an observation of the solution in some time $T_0 > 0$, as usual in the parabolic case. To adapt this point to our case, we introduced Assumption $\mathfrak{I}$, which will allow us to apply the Carleman estimate and carry out the method.

We will also need the following slight modification of Theorem 1.1. Let $\omega_0 \Subset \omega$ be non-empty and open, $\kappa \in (1,2)$ and β be constructed by Lemma 3.1 with ω_0 as before. A Carleman estimate on $Q := (-T,T) \times (0,L)$ like the one in Theorem 1.1 can be derived just by modifying the weights η and ξ as follows:

$$\eta(t,x) = \frac{e^{\kappa\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)}}{(t+T)(T-t)} \quad \text{and} \quad \xi(t,x) = \frac{e^{\lambda\beta(x)}}{(t+T)(T-t)}$$

for $(t,x) \in Q$. More precisely, we have the following.

Proposition 5.1. *Let η and ξ be as previously defined. Under Hypothesis $\mathfrak{M}$, there exist $s_0 > 0$, $\lambda_0 > 0$ and a constant $C > 0$ depending on ω, Γ, L, T, p, $\|\beta\|_{C^3([0,L]\setminus\Gamma)}$, s_0 and λ_0 such that for any $u \in \mathcal{V}$ we have*

$$C\iint_Q e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2 + s^3\lambda^4\xi^3|u_x|^2 + s\lambda^2\xi|u_{xx}|^2)dxdt \le \|e^{-s\eta}\mathcal{L}u\|^2_{L^2(Q)}$$

$$+\iint_{(-T,T)\times\omega} e^{-2s\eta}(s^5\lambda^6\xi^5|u|^2dxdt + s^3\lambda^4\xi^3|u_x|^2dxdt + s\lambda^2\xi|u_{xx}|^2)dxdt \quad (5.1)$$

for any $s \geq s_0$ *and* $\lambda \geq \lambda_0$*, with* $\mathcal{L}$ *and* $\mathcal{V}$ *similarly defined as in* (1.3)*.*

With this inequality at hand, we can prove the Lipschitz stability result.

Proof of Theorem 1.3. The existence of solutions for initial data $y_0 \in H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L)$ can be established following the framework of Bona, Sun and Zhang [BSZ03, Theorem 4.1]. First of all, the addition of the term $\mu(x)y_x$ on the linear part is easily handled by a fixed-point argument as in Proposition 2.6. Second, we need to look at the equation satisfied by $w = y_{tt}$ and then, similar arguments as in Proposition 2.6 can be used to establish the needed estimates and to prove that for $y_0 \in H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L)$, the corresponding solution y belongs to $C([0,T], H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L)) \cap L^2(0,T; H^7_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L))$. By using classical Sobolev embedding on each I_k, this ensures the regularity needed to employ the Bukhgeĭm-Klibanov method.

Let us consider two coefficients $\mu = \mu(x)$ and $\nu = \nu(x)$ belonging to $\mathfrak{P}^{sym}_{\leq m}(0,L)$ with the corresponding solutions $y := y[\mu]$ and $z := z[\nu]$ of (1.10) with the same main coefficient p and initial condition $y_0 \in H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L)$. Let us define

$$u(t,x) := y(t,x) - z(t,x) \quad \text{and} \quad \sigma(x) := \nu(x) - \mu(x).$$

Step 1: auxiliary system. Let $v = u_t$ and note that $v_0(x) := \sigma(x)y_0'(x)$ satisfies $v_0(x) = v_0(L-x)$ for all $x \in [0,L]$. The system satisfied by $\psi := \widehat{v}$ is

$$\begin{cases} \psi_t + p(x)\psi_{xxx} + (1+\widehat{z})\psi_x + \widehat{y}_x\psi = \check{f}, & (t,x) \in (-T,T)\times(0,L), \\ \psi(t,0) = \psi(t,L) = 0, & t \in (-T,T), \\ \psi_x(t,L) = 0, & t \in (0,T), \\ \psi_x(t,L) = -v_x(0,-t), & t \in (-T,0), \\ \psi(0,x) = \sigma(x)y_0'(x), & x \in (0,L), \end{cases}$$

coupled by the corresponding transmission conditions (TC) with coefficient p, with $f := \sigma(x)z_{xt} - y_{xt}u - z_tu_x$ and the symmetric and anti-symmetric extensions being defined, respectively, as

$$\widehat{g}(t,x) = \begin{cases} g(t,x), & (t,x) \in [0,T]\times[0,L], \\ g(t,L-x), & (t,x) \in [-T,0)\times[0,L], \end{cases}$$

$$\check{g}(t,x) = \begin{cases} g(t,x), & (t,x) \in [0,T]\times[0,L], \\ -g(-t,L-x), & (t,x) \in [-T,0)\times[0,L]. \end{cases}$$

Step 2: First use of the Carleman estimate. By compactness, we can find $\omega_0 \Subset \omega$ which is symmetric with respect to $L/2$ and (ω_0, p) satisfies Hypothesis $\mathfrak{M}$. Let $K > 0$ be some constant such that

$$\max\{\|y\|_{W^{1,\infty}(0,T;W^{1,\infty}(0,L))}, \|z\|_{W^{1,\infty}(0,T;W^{1,\infty}(0,L))}\} \leq K. \quad (5.2)$$

This is consistent given that $y, z \in C([0,T], H^6_\Gamma(0,L) \cap \mathcal{H}^3_\Gamma(0,L))$ and by classical Sobolev embedding $H^1(I_k) \hookrightarrow L^\infty(I_k)$ applied on each $k \in [\![0, N-1]\!]$. We shall focus on the following integral

$$\int_{-T}^0 \int_0^L \xi^2 w \mathcal{L}_1 w dxdt = \frac{1}{2}\int_0^L \xi^2(0,x)|w(0,x)|^2dx + \mathcal{J}, \quad (5.3)$$

where $\mathcal{J}$ can be estimated by the Carleman estimate for the conjugated operator (obtained throughout the proof of Proposition 5.1, compare with Proposition 3.3) as follows

$$|\mathcal{J}| \leq Cs^{-3}\lambda^{-3}\left(\iint e^{-2s\eta}|\check{f}|^2dxdt + \iint_{(-T,T)\times\omega_0}(s^5\lambda^6\xi^5|w|^2dxdt + s^3\lambda^4\xi^3|w_x|^2dxdt + s\lambda^2\xi|w_{xx}|^2)dxdt\right). \quad (5.4)$$

Since $|y_0'(x)| \geq r_0 > 0$, we get

$$\int_0^L \xi^2(0,x)|w(0,x)|^2dx = \int_0^L e^{-2s\eta(0,x)}\xi^2(0,x)|\sigma(x)y_0'(x)|^2dx$$
$$\geq r_0^2 \int_0^L e^{-2s\eta(0,x)}\xi^2(0,x)|\sigma(x)|^2dx.$$

Thus, we can use the last inequality to get a bound by below from identity (5.3) and then use (5.4) along with Young's inequality to get a bound by above, resulting in

$$\int_0^L e^{-2s\eta(0,x)}\xi^2(0,x)|\sigma(x)|^2dx \lesssim s^{-5/2}\lambda^{-3}\Bigg(\iint e^{-2s\eta}|\check{f}|^2dxdt$$
$$+\iint_{(-T,T)\times\omega_0}(s^5\lambda^6\xi^5|w|^2dxdt + s^3\lambda^4\xi^3|w_x|^2dxdt + s\lambda^2\xi|w_{xx}|^2)dxdt\Bigg)$$

for $s \geq s_0$ and $\lambda \geq \lambda_0$. using the fact that η is even in time and $w = e^{-2s\eta}\psi = e^{-2s\eta}\widehat{v}$ we have

$$\int_0^L e^{-2s\eta(0,x)}\xi^2(0,x)|\sigma(x)|^2dx \;\lesssim\; s^{-5/2}\lambda^{-3}\left(\iint_Q (e^{-2s\eta(t,x)} + e^{-2s\eta(t,L-x)})|f|^2dxdt + \mathcal{M}_{\omega_0}(v)\right) \tag{5.5}$$

where, $\mathcal{M}_{\omega_0}(v)$ gathers the local terms in v as follows

$$\mathcal{M}_{\omega_0}(v) := \iint_{(0,T)\times\omega_0}(s^5\boldsymbol{\xi}_5e^{-2s\eta}|v|^2 + s^3\boldsymbol{\xi}_3e^{-2s\eta}|v_x|^2 + s\boldsymbol{\xi}_1e^{-2s\eta}|v_{xx}|^2)dxdt, \tag{5.6}$$

with $\boldsymbol{\xi}_k$ defined as $\boldsymbol{\xi}_k(t,x) := \xi^k(t,x) + \xi^k(t,L-x)$, $(t,x) \in Q$, for $k = 1,3,5$. Observe that we just used the change of variables $x \mapsto L - x$ and that ω_0 is symmetric with respect to $L/2$.

We now look at the terms involving f. By using the bound (5.2), we get

$$\iint_Q e^{-2s\eta(t,x)}|f|^2dxdt = \iint_Q e^{-2s\eta(t,x)}|\sigma(x)z_{xt} - y_{xt}u - z_tu_x|^2dxdt$$
$$\lesssim \int_0^L e^{-2s\eta(0,x)}|\sigma(x)|^2dx + \int_0^T\int_0^L e^{-2s\eta}(|u|^2 + |u_x|^2)dxdt.$$

Similarly, we have

$$\iint_Q e^{-2s\eta(t,L-x)}|f|^2dxdt = \iint_Q e^{-2s\eta(t,L-x)}|\sigma(x)z_{xt} - y_{xt}u - z_tu_x|^2dxdt$$
$$\lesssim \int_0^L e^{-2s\eta(0,L-x)}|\sigma(x)|^2dx + \int_0^T\int_0^L e^{-2s\eta(t,L-x)}(|u|^2 + |u_x|^2)dxdt$$
$$= \int_0^L e^{-2s\eta(0,x)}|\sigma(x)|^2dx + \int_{-T}^0\int_0^L e^{-2s\eta(t,x)}(|\widehat{u}|^2 + |\widehat{u}_x|^2)dxdt,$$

where we used that $t \in [0,T) \mapsto e^{-s\eta(t,L-x)}$ is decreasing for any $x \in [0,L]$ and the change of variables $x \mapsto L - x$. Gathering the last two inequalities we obtain

$$\iint_Q (e^{-2s\eta(t,x)} + e^{-2s\eta(t,L-x)})|f|^2dxdt \lesssim \int_0^L e^{-2s\eta(0,x)}|\sigma(x)|^2dx + \int_{-T}^T\int_0^L e^{-2s\eta}(|\widehat{u}|^2 + |\widehat{u}_x|^2)dxdt. \tag{5.7}$$

It remains to estimate the second term on the right-hand side of the above inequality.
Step 3: Second use of the Carleman estimate. We apply the Carleman estimate (5.1) to the equation satisfied by $\widehat{u}$ to obtain

$$\int_{-T}^{T}\int_0^L e^{-2s\eta(t,x)}(|\widehat{u}|^2+|\widehat{u}_x|^2)dxdt \lesssim s^{-3}\lambda^{-4}\left(\int_{-T}^{T}\int_0^L e^{-2s\eta}|\sigma z_{xt}|^2dxdt \right.$$
$$\left. + \iint_{(-T,T)\times\omega_0} e^{-2s\eta}(s^5\xi^5|\widehat{u}|^2dxdt + s^3\xi^3|\widehat{u}_x|^2dxdt + s\xi|\widehat{u}_{xx}|^2)dxdt\right).$$

As before, we use the bound (5.2) to bound the first term. Then, we use the definition of the symmetric extension to we rewrite the local terms at the right-hand side of the last inequality as integrals over $(0,T)\times\omega_0$, Thus, with $\mathcal{M}_{\omega_0}(u)$ as defined in (5.6), we get

$$\int_{-T}^{T}\int_0^L e^{-2s\eta(t,x)}(|\widehat{u}|^2+|\widehat{u}_x|^2)dxdt \lesssim s^{-3}\lambda^{-4}\left(\int_0^L e^{-2s\eta(0,x)}|\sigma(x)|^2dx + \mathcal{M}_{\omega_0}(u)\right). \tag{5.8}$$

Putting together inequalities (5.5)-(5.7)-(5.8) and then using that $x\in[0,L]\mapsto e^{-2s\eta(0,x)}\xi^2(0,x)$ is positively bounded by below, we obtain that

$$\left(1-(s^{-11/2}\lambda^{-7}+s^{-5/2}\lambda^{-3})\right)\int_0^L|\sigma(x)|^2dx \lesssim (s^{-5/2}\lambda^{-3}\mathcal{M}_{\omega_0}(v)+s^{-3}\lambda^{-4}\mathcal{M}_{\omega_0}(u)) \tag{5.9}$$

for any $s\geq s_0$ and $\lambda\geq\lambda_0$. Noticing that $(t,x)\in(0,T)\times\omega_0\mapsto s^k\boldsymbol{\xi}_k(t,x)e^{-2s\eta(t,x)}$ is bounded by above for $k=1,3,5$, we readily get

$$\mathcal{M}_{\omega_0}(u)+\mathcal{M}_{\omega_0}(v)\lesssim \|y-z\|_{H^1(0,T;H^2(\omega_0))}.$$

Since $\sigma=\mu-\nu$ and $\omega_0\Subset\omega$, the proof ends by choosing s and λ large enough in (5.9) so that the left-hand side is made positive. ☐

Remark 5.2. We point out that the regularity assumption $y_0\in H^6_\Gamma(0,L)\cap\mathcal{H}^3_\Gamma(0,L)$ is not sharp. From the proof, we need that z_{xt}, y_{xt} belong to $L^\infty(Q)$, which would follow by Sobolev embedding provided they both belong to $L^\infty([0,T],H^s(0,L))$, with $s>1/2$. This could be achieved if we ask $y_0\in H^{4+s}_\Gamma(0,L)\cap\mathcal{H}^3_\Gamma(0,L)$. Nevertheless, a rigorous proof will employ Tartar's nonlinear interpolation (see [BSZ03, Section 4]) and a characterization for interpolation of spaces involving the transmission conditions (2.2). The latter is most likely to be true, but up to the author's knowledge, there is no straightforward proof of such a fact available in the current literature. We do not deepen in this direction as it is outside of the scope of this work.

6. Some further remarks

6.1. **Boundary observability.** Under the hypothesis $p_k>p_{k-1}$ with $k\in[\![0,N-1]\!]$, a straightforward modification to the proof of Lemma 3.1 lead us to the construction of β with *observation at* $x=0$. Given $\lambda>0$, we define

$$\eta(t,x)=\frac{e^{\kappa\lambda\|\beta\|_\infty}-e^{\lambda\beta(x)}}{t(T-t)} \quad\text{and}\quad \xi(t,x)=\frac{e^{\lambda\beta(x)}}{t(T-t)}, \tag{6.1}$$

for all $(t,x)\in Q$ and some $\kappa\in(1,2)$. By following the same steps as before, we can obtain a Carleman estimate with boundary observation for the solutions of the system

$$\begin{cases} \varphi_t+p(x)\varphi_{xxx}+\varphi_x=0, & (t,x)\in(0,T)\times(0,L),\\ \varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0, & t\in(0,T),\\ \varphi(T,x)=\varphi_T(x), & x\in(0,L),\end{cases} \tag{6.2}$$

coupled by the usual transmission conditions (TC). The Carleman estimate is the following.

Proposition 6.1. *Let η and ξ be the weight functions defined by (6.1). Suppose that $p_k > p_{k-1}$ for all $k \in [\![1, N-1]\!]$. Then there exist $s_0 > 0$, $\lambda_0 > 0$ and a constant $C > 0$ depending on L, T, ρ_0, ρ_1, $\|\beta\|_{C^3}$, r, s_0 and λ_0 such that for all $\varphi_T \in \mathcal{D}(\mathcal{A}^*)$ we have*

$$C \iint_Q e^{-2s\eta}(s^5\lambda^6\xi^5|\varphi|^2 + s^3\lambda^4\xi^3|\varphi_x|^2 + s\lambda^2\xi|\varphi_{xx}|^2)dxdt \leq s\lambda \int_0^T e^{-2s\eta(t,0)}\xi(t,0)|\varphi_{xx}(t,0)|^2 dt,$$

for any $s \geq s_0$ and $\lambda \geq \lambda_0$, where φ is the solution of (6.2) associated to φ_T.

As classically done by the HUM method, under the hypothesis that $p_k > p_{k-1}$ for $k \in [\![1, N-1]\!]$, the previous Carleman estimate can be combined with a dissipation estimate to obtain, for instance, the boundary null controllability of the linear KdV equation

$$\begin{cases} y_t + p(x)y_{xxx} + y_x = 0, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = h(t),\ y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases}$$

coupled by the transmission conditions (TC), with control $h \in L^2(0,T)$ and $y_0 \in L^2(0,L)$.

In regards to the boundary control to the trajectories for the constant main coefficient case, Glass and Guerrero [GG08] provided a positive result with one control acting on the left point of the interval. Trying to adapt their ideas to the discontinuous case is an interesting problem, as the proof is more involved, mainly due to the several regularity technicalities related to the boundary value problem and also that it heavily uses interpolation arguments. The latter is a problem in itself in the case of discontinuous coefficients, see Remark 2.11. We point out that similar issues has been faced in Parada [Par24] when studying the KdV equation on a star-shaped network.

Additionally, for the problem of exact controllability when only one control is acting on the boundary, one may expect to see the critical length phenomena in the discontinuous setting as well. Thus, the problem of exact controllability for every time, for a set of lengths and length of the set of discontinuities is a wide open problem. See [Cré16, Proposition 4, Remark 2] where the control is acting on the Neumann boundary condition.

6.2. **Monotonicity hypothesis on the Carleman estimate.** The monotonicity hypothesis on p, enforced trough Assumption $\mathfrak{M}$, is crucial to obtain the Carleman estimate with main piecewise constant coefficient. Indeed, we can then construct a weight function which is piecewise monotone and satisfies the same transmission conditions as given by the main PDE under consideration, which further allows us to obtain a weighted norm for the trace terms at the interface.

Regarding applications to inverse problems, it is worth noticing that a similar Carleman estimate with boundary observation as the one used in [BCCM14] can be obtained under the hypothesis $p_k > p_{k-1}$, $k \in [\![1, N-1]\!]$ (see also Proposition 6.1). However, this monotonicity condition is not compatible with Hypothesis $\mathfrak{M}$, the latter being necessary to employ the reflection trick and therefore to avoid observations in some time $T_0 \in (0,T)$, as commonly found in the parabolic case. Whether one can get rid of these monotonicity hypothesis on the coefficient p is an open problem.

The main difficulty is to construct a weight function that allows us to estimate the interface terms coming from I_{12} in the Carleman estimate. These terms are not necessarily zero if the weight function does not satisfy the transmission conditions. Observe that a similar difficulty is faced when establishing a Carleman estimate for the KdV equation under Colin-Ghidaglia boundary conditions, see Guilleron [Gui14] and Carreño and Guerrero [CG18].

Appendix A.

A.1. **Inequalities toolbox.** Let $I \subset \mathbb{R}$ be a non-empty interval and $T > 0$. Let us introduce for $s \geq 0$ the Banach space

$$\boldsymbol{X}_T^s(I) = C([0,T], L^2(I)) \cap L^2(0,T; H^{s+1}(I)),$$

equipped with the natural norm. We have the following Lemma, used in Section 2 to obtain the well-posedness of the nonlinear system (1.1); see Proposition 2.6.

Lemma A1. [BSZ03, Lemma 3.3] *Let $s \geq 0$ be given. There exists $C > 0$ such that for any $T > 0$ and u, $v \in \boldsymbol{X}^s_T(I)$,*

$$\int_0^T \|u(t,\cdot)v_x(t,\cdot)\|_{H^s(I)}dt \leq C(T^{1/2} + T^{1/3})\|u\|_{\boldsymbol{X}^s_T(I)}\|v\|_{\boldsymbol{X}^s_T(I)}.$$

The following result allows us to consider yy_x as a source term in (2.3)-(TC).

Proposition A2. [Ros97, Proposition 4.1] *Let $y \in L^2(0,T;H^1(I))$. Then $yy_x \in L^1(0,T;H^1(I))$ and the map*

$$y \in L^2(0,T;H^1(I)) \longmapsto yy_x \in L^1(0,T;L^2(I))$$

is continuous and there exists $C > 0$ such that

$$\|yy_x - zz_x\|_{L^1(0,T;L^2(I))} \leq C\big(\|y\|_{L^2(0,T;H^1(I))} + \|z\|_{L^2(0,T;H^1(I))}\big)\|y - z\|_{L^2(0,T;H^1(I))}.$$

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