STABILIZATION AND CONTROL OF THE NONLINEAR PLATE EQUATION

CRISTÓBAL LOYOLA

ABSTRACT. In this article we prove semiglobal stabilization and exact controllability results for nonlinear plate equations with hinged boundary conditions and analytic nonlinearity. These results hold when the damping or control is localized in a region where observability for the linear Schrödinger equation is known to hold. At the core of these results lies a new unique continuation property for the nonlinear plate equation, which significantly relaxes the geometric conditions required for such property to hold. This property is obtained by combining recent results on propagation of analyticity in time and unique continuation for linear plate operators. More broadly, our approach exploits the linear observability of the plate equation to establish both stabilization and control results. First, we prove exponential decay of the nonlinear energy under a defocusing assumption on the nonlinearity. Second, under a weaker asymptotic assumption on the nonlinearity, we prove semiglobal exact control by analyzing control properties inside the compact attractor provided by the dynamics of the damped equation.

CONTENTS

1. INTRODUCTION

Let $(\mathcal{M}, g)$ be a smooth compact connected Riemannian manifold of dimension $d \leq 3$ with (possibly empty) smooth boundary $\partial\mathcal{M}$. In this article we consider the following nonlinear plate equation

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = 0 & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial\mathcal{M}, \\ (u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M}, \end{cases} \tag{1.1}$$

where $(u_0, u_1) \in \left(H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})\right) \cap L^2(\mathcal{M})$ and equipped with hinged boundary conditions whenever $\partial\mathcal{M} \neq \emptyset$. Here the nonlinearity $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is assumed to be smooth in the first variable and analytic in the second one. Furthermore, depending on the situation we consider, it will satisfy assumptions that constrain the dynamics of (1.1). In the model, the constant $\beta \geq 0$ is so that we have the Poincaré-like inequality, we thus may require $\beta > 0$ when $\partial\mathcal{M} = \emptyset$.

Models of the form (1.1) are Kirchhoff–Love thin-plate equations with a local nonlinearity, describing the transverse deflection of a thin plate and capturing material or contact effects. In this article, we study internal stabilization and internal control for (1.1) when the system is acted upon by either a localized damping term (interpreted as local frictional damping) or a localized control input (interpreted as a distributed transverse load). We refer to [LL88, Lag89] for further details on the phenomenology of such systems, as well as for early results on control and stabilization. The validity of our results holds whenever the damping or control is supported in a nonempty open set $\omega \subset \mathcal{M}$ satisfyinh the following assumption:

Assumption 1. The Schrödinger equation is observable in L^2 from ω in time $T > 0$: there exists $C = C(T, \omega) > 0$ such that for any $v_0 \in L^2(\mathcal{M})$ it holds

$$\|v_0\|_{L^2(\mathcal{M})}^2 \le C \int_0^T \|\mathbb{1}_\omega e^{it\Delta_g} v_0\|_{L^2(\mathcal{M})}^2 dt.$$

We point out right away that the sharp geometric condition on ω necessary for the observability of the Schrödinger equation remains an open question. Yet, it has been the object of many investigations. In the following situations, Assumption 1 is known to be true for any $T > 0$:

(1) $(\mathcal{M}, g)$ is a compact Riemannian manifold with or without boundary and ω satisfies the *Geometric Control Condition*. See Lebeau [Leb92].

(2) $(\mathcal{M}, g) = ((0, 1)^d, \text{Euclid})$ with $d \le 3$, ω is any nonempty open set. This was first proved by Jaffard for $d = 2$ [Jaf90] and Komornik [Kom92] for other dimensions (actually directly for the beam equation). Other proofs have also been given later by Burq-Zworski [BZ04], Anantharaman-Macià [AM14]. Note here that the proofs are given for the torus $\mathbb{T}^d$, but an easy argument of symmetrization allows to recover the same result for the Dirichlet boundary condition.

(3) $(\mathcal{M}, g) = (\mathbb{D}, \text{Euclid})$ is the Euclidean closed disk in $\mathbb{R}^2$ and $\omega \cap \partial\mathcal{M} \ne \emptyset$. See Anantharaman, Léautaud and Macià [ALM16, Theorem 1.2],

(4) $(\mathcal{M}, g)$ is the Bunimovich stadium and ω controls geometrically $\mathcal{M} \setminus R$, R being the rectangular part, see Burq-Zworski [BZ04, Theorem 9].

(5) $(\mathcal{M}, g)$ is a compact, boundaryless connected Riemannian surface whose flow has the Anosov property, ω is any nonempty open set. See Dyatlov-Jin-Nonnenmacher [DJN22, Theorem 5].

(6) $(\mathcal{M}, g)$ is a compact, boundaryless Riemannian manifold of dimension d and constant curvature $\equiv -1$, and the observation $\mathcal{C}\psi = a\psi$ is made through a smooth function a on M such that the set $\{\rho \in S^*\mathcal{M} \mid a^2(\Phi_t(\rho)) = 0, \ \forall t \in \mathbb{R}\}$ has Hausdorff dimension $< d$. Here Φ_t is the bicharacteristic flow on $T^*\mathcal{M}$. See Anantharaman-Rivière [AR12, Theorem 2.5].

Remark 1.1. Observability of linear Schrödinger equations in unbounded domains has also been object of several studies, we refer to Prouff [Pro25] and the references therein.

Compared to the current literature on stabilization and control of nonlinear plates like (1.1), our results widely improve the geometric conditions for which such results hold true. In this direction, we point out that most available results assume some sort of multiplier-type condition on the damping or control zone, which is known to be stronger than, for instance, the Geometric Control Condition [Mil03]. Let us explain the origin of our assumption and the difficulties previously encountered in the literature.

To fix ideas, consider the stabilization problem with a defocusing nonlinearity. It is well known that the decay of the energy of the damped system (see (1.2) below) is equivalent to proving an observability inequality for the nonlinear equation. In our situation, this compactness-uniqueness strategy can be roughly summarized as follows:

linear observability + unique continuation $\Longrightarrow$ nonlinear observability.

Although not explicitly, this scheme was introduced by Dehman, Lebeau and Zuazua [DLZ03] to stabilize the semilinear defocusing wave equation in $\mathbb{R}^3$ with damping active outside a ball. Furthermore, this strategy has been successfully employed for both the wave and Schrödinger equations in different geometric settings [DGL06, Lau10, JL13, JL20, Per23]. When it comes to our case, as it has already been noticed by several authors, the plate operator with hinged boundary conditions can be decomposed into two Schrödinger-type operators $\partial_t^2 + \Delta_g^2 = (i\partial_t - \Delta_g)(i\partial_t + \Delta_g)$, showing that both operators might share similar properties. In this direction, the link between the linear observability of the Schrödinger and the plate equation is now rather well-understood after Lebeau [Leb92], who introduced a rather general strategy to transfer observability inequalities from the Schrödinger to the plate equation, see [LL24, Section 4]. See also Miller [Mil12] for the connection between the observability of linear waves and Schrödinger-like equations. Here, as done under the GCC in the context of the wave and Schrödinger equations, we perform the propagation arguments by exploiting the linear observability inequality along with the compactness properties of the non-linearity, the latter being inherited from the well-posedness framework. When it comes to unique continuation for the nonlinear plate equation, the situation is far less understood and most of the available results in the literature hold under rather restrictive geometric conditions, as they are derived from Carleman-estimates-based strategies. In this article we prove unique continuation for the nonlinear plate equation by combining recent results on unique continuation for linear plate operators by Filippas, Laurent, and Léautaud [FLL25] and propagation of analyticity in time from Laurent and the author [LL24]. We can roughly state that this strategy can be carried out whenever observability for the linear equation holds and the nonlinearity is analytic and enjoys some compactness property, hence explaining Assumption 1. In this spirit, new observability results for the Schrödinger equation in bounded domains, will most likely automatically widen the validity of our results.

Regarding semiglobal control, by time reversibility of the equation, it is known that it can be deduced by combining stabilization of the damped equation along with nonlinear local control around 0. If we instead relax the defocusing assumption to an asymptotic defocusing condition, the long-time dynamics of the equation become more complicated and the previous strategy does not work as the damped plate may have nontrivial equilibria. However, following Joly and Laurent [JL14], we can use the unique continuation property to prove that the damped plate generates a gradient system and admits a compact global attractor. From this fact, a control strategy can be implemented to prove semiglobal control by studying the controllability properties inside this compact attractor.

1.1. **Main results.**

We describe our main results below. First, we state the global stabilization result whenever a localized damping term $+\gamma^2(x)\partial_t u$ is added to (1.1) and the nonlinearity is assumed to be analytic and defocusing. When such a defocusing assumption is relaxed by a similar asymptotic condition, we can prove the semiglobal control of the system by studying the attractor of the damped plate combined with a local control result around equilibria. Finally, we describe the unique continuation result for (1.1) whenever the solution is observed to be zero in a zone from where Assumption 1 holds.

1.1.1. *Semiglobal stabilization.*

Let us consider the following nonlinear plate with localized damping

$$
\begin{cases}
\partial_t^2 u + \Delta_g^2 u + \beta u + \gamma^2(x)\partial_t u + f(x, u) = 0 & (t, x) \in [0, +\infty) \times \text{Int}(\mathcal{M}), \\
u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & (t, x) \in [0, +\infty) \times \partial\mathcal{M}, \\
(u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M},
\end{cases}
\tag{1.2}
$$

where $\gamma \in C^\infty(\mathcal{M})$ satisfies $\gamma(x) \geq \gamma_0 > 0$ for $x \in \omega$. Here we assume that $f(\cdot, 0) = 0$ and that it satisfies the defocusing assumption

$$\begin{aligned} sf(x,s) &\geq 0 && \text{if } \partial\mathcal{M} \neq \emptyset, \\ sf(x,s) &\geq \alpha s^2 && \text{if } \partial\mathcal{M} = \emptyset, \end{aligned} \tag{1.3}$$

for some $\alpha > 0$ and for all $(x, s) \in \mathcal{M} \times \mathbb{R}$.

The nonlinear energy $E \in C^0(H_D^2 \times L^2(\mathcal{M}), \mathbb{R}^+)$ associated to (1.2) is given by

$$E(u, \partial_t u) := \frac{1}{2} \int_{\mathcal{M}} (|\partial_t u|^2 + |\Delta u|^2 + \beta |u|^2) dx + \int_{\mathcal{M}} V(x, u) dx,$$

where $V(x, u) = \int_0^u f(x, s) ds$. Since $d \leq 3$, due to Sobolev embedding the energy E is well-defined. Moreover, if u solves (1.2), formally we have the identity

$$\partial_t E(u, \partial_t u) = -\int_{\mathcal{M}} \gamma^2(x) |\partial_t u|^2 dx \leq 0, \tag{1.4}$$

which tell us that the energy E is non-increasing along the trajectories. We are interested in the following property:

(ED) for any $E_0 \geq 0$, there exist $C > 0$ and $\lambda > 0$ such that, for any solution u of (1.2) with $E(U(0)) \leq E_0$, it holds $E(U(t)) \leq Ce^{-\lambda t}E(U(0))$.

This property means that the damping term $+\gamma\partial_t u$ stabilizes any solution of (1.2) to 0 at an exponential rate which depends on the size of the data. Our first main theorem is the following.

Theorem 1.2. *Assume that ω satisfy Assumption 1. If f is real analytic in the second variable and it satifies (1.3), then the exponential decay property **(ED)** holds.*

The previous result follows by proving an appropriate observability inequality for the nonlinear energy of (1.2), see Theorem 2.8 below. As we already pointed out, we obtain the desired observability by a compactness-uniqueness method, where we exploit the linear observability of the plate equation and the unique continuation property for nonlinear plates. With this strategy, the constant C and the decay rate λ of the nonlinear energy both depend on the size of the initial data R_0. However, it might be possible to get the decay rate to be uniform with respect to the size of the data: we know that solutions are small for large time, therefore the nonlinear term could be neglected yielding almost the same decay rate as in the linear equation. See for instance [LRZ10] for the case of the KdV equation. The possibility of dropping the dependence of C with respect to R_0 is an open problem. We refer to Le Balc'h and Martin [BM25], where uniform global stabilization for the subcritical Schrödinger equation on the torus was obtained by a direct method using nonlinear Carleman estimates and Morawetz-type estimates.

1.1.2. *Dynamics and controllability.* Let us now introduce the following controlled system

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = \mathbb{1}_{\widetilde\omega} g & (t, x) \in [0, T] \times \text{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial\mathcal{M}, \\ (u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M}, \end{cases} \tag{1.5}$$

with $g \in L^\infty([0, T], L^2(\mathcal{M}))$ and $\widetilde\omega \subset \mathcal{M}$ some nonempty open set. In this section, we relax (1.3) by an asymptotic assumption on the nonlinearity, that is, we assume that there exists $R > 0$ such that

$$f(x, 0) = 0 \text{ for every } x \in \partial\mathcal{M} \quad \text{and} \quad sf(x, s) \geq 0 \text{ for every } (x, s) \in \mathcal{M} \times \{|s| \geq R\}. \tag{1.6}$$

The controllability result is the following.

Theorem 1.3. *Let f be analytic in the second variable and satisfy condition (1.6). Let $\omega \Subset \widetilde{\omega}$ be two nonempty open sets such that ω satisfies Assumption 1 in time $T_0 > 0$. For any bounded set $\mathcal{B} \subset H_D^2 \times L^2(\mathcal{M})$ there exists $T(\mathcal{B}) > 0$ such that for any (u_0, u_1) and $(\widetilde{u}_0, \widetilde{u}_1)$ in $\mathcal{B}$, there exist $T \in (0, T_{\mathcal{B}})$ and a control $g \in L^\infty([0,T], L^2(\mathcal{M}))$ supported in $(0,T) \times \widetilde{\omega}$ such that the unique solution $(u, \partial_t u) \in C^0([0,T], H_D^2 \times L^2(\mathcal{M}))$ of (1.5) satisfies $(u(T), \partial_t u(T)) = (\widetilde{u}_0, \widetilde{u}_1)$.*

A well-known strategy to obtain semiglobal control in the defocusing case with $f(\cdot, 0) = 0$, as before, is based on the time reversibility of the equation. Indeed, it combines the decay of the energy with a local control to achieve the target 0, which in turn allows us to reach any target by time reversibility. Note, however, that the asymptotic condition (1.6) is weaker than the defocusing condition (1.3), and thus the long-time dynamics of the equation is more complicated, allowing the existence of nontrivial equilibria. Following [JL14], it is possible to implement a control strategy in the presence of more involved dynamics, as a consequence of (1.6). In this direction, the previous control result relies on the existence and description of a global attractor for the damped system (1.2) and on a local exact control property around an equilibrium.

Theorem 1.4. *Let f be analytic in the second variable and satisfy condition (1.6). If ω satisfies Assumption 1 then the dynamical system generated by the damped plate equation (1.2) in $H_D^2 \times L^2(\mathcal{M})$ is gradient and admits a compact global attractor $\mathcal{A}$.*

The main idea is that under assumption (1.6), solutions of the damped plate (1.2) arrive close to the attractor $\mathcal{A}$ for large time. Then, by combining local control around equilibrium points and using heteroclinic connections to travel from one equilibrium to another within the attractor, we can still obtain global controllability by using the time reversibility of the system. We refer to Raugel [Rau02] for concepts and main results on dynamical systems of PDEs. We have gathered the ones that will be employed in this article in Section B.

Few results in the literature address the global exact controllability of nonlinear plate equations, among which we mention the following. Lasiecka and Triggiani [LT91] considered globally Lipschitz nonlinearities with a control acting on the entire boundary, relying on linear results and a suitable global inversion theorem. Zhang [Zha01] studied a nonlinearity with superlinear growth, where the analysis relied on deriving suitable observability estimate for a plate equation with potential via Carleman estimates. Eller and Toundykov [ET15] examined several semilinear plate models with an internal control acting on a collar neighborhood of the boundary, with their method relying partly on uniform stabilization through a suitable feedback. For a comprehensive summary of results on nonlinear plates up to 2015, we refer the reader to the that article.

1.1.3. *Unique continuation property.* As it was already explained, a key property to establish the aforementioned results is the one of unique continuation for solutions of (1.1).

Theorem 1.5. *Let f be real analytic in the second variable. Let $0 < T_0 < T$ and $\omega \Subset \widetilde{\omega}$ are two nonempty open sets such that ω satisfies Assumption 1 in time T_0. If one solution $U = (u, \partial_t u) \in C^0([0,T], H_D^2 \times L^2(\mathcal{M}))$ of (1.1) satisfies $\partial_t u = 0$ in $[0,T] \times \widetilde{\omega}$, then $\partial_t u = 0$ in $[0,T] \times \mathcal{M}$ and u is an equilibrium point of (1.1), that is, solution of*

$$\begin{cases} \Delta_g^2 u + \beta u + f(x, u) = 0 & x \in \mathrm{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & x \in \partial\mathcal{M}. \end{cases} \tag{1.7}$$

If, moreover, the nonlinearity f satisfies the defocusing assumption (1.3), then $u \equiv 0$.

To our knowledge, most available unique continuation results for linear plates mostly rely on Carleman estimates, requiring strong geometrical assumptions related to pseudoconvexity but low regularity on the coefficients, see for instance Isakov [Isa97]. See also [LTY03, Wan07, ET15, IM25] and the references therein for works addressing Carleman estimates for plate equations with different boundary conditions. In this direction, Filippas, Laurent, and Léautaud [FLL25] obtained a unique

continuation result for the linear plate operator with lower-order terms assuming Gevrey-2 regularity in time and allowing fairly general geometries. This has to be compared with the remarkable theorem of Tataru [Tat95, Tat99], Robbiano-Zuily [RZ98] and Hörmander [Hör97], which notably does not apply to the plate operator [FLL25, Appendix B].

When moving towards nonlinear results of this kind, very few results are available, and those mostly rely on results for the linear equation. Here, we hinge on a recent result due to Laurent and the author [LL24, Theorem 1.6], in which the nonlinear structure of the equation plays a crucial role to propagate analyticity in time from the vanishing zone to the full solution. This allows us to achieve the desired regularity needed to apply the aforementioned unique continuation result for linear plates. Hence, Theorem 1.5 substantially improves upon the existing literature on unique continuation for nonlinear plates, with respect to the geometric conditions under which such a property holds.

Remark 1.6. The results presented so far are most likely to be true in higher dimensions $d \geq 4$ under further assumptions on the growth of nonlinearity f (and its derivative). For instance, in dimension $d = 4$, since we are no longer able to use the embedding $H_D^2 \hookrightarrow L^\infty$ we might need to assume polynomial growth on f. In this way, the nonlinear energy could be controlled by the linear energy and the nonlinearity might still enjoy the needed compactness properties, similar to Theorem 2.4 below. For $d \geq 5$, the situation becomes more complicated as it may require the use of appropriate Strichartz estimates, which, To our knowledge are not available for plates equation in the geometries we are concerned with. We mention Pausader [Pau07] for Strichartz estimates in the euclidean space $\mathbb{R}^d$.

1.1.4. *Brief literature.* Given the extensive literature on the topic, we mainly focus on results concerning nonlinear models. More precisely, for stabilization and control results on plates, our main references are Eller and Toundykov [ET15] and Tucsnak, Bournissou and Ervedoza [TBE24], as well as the references therein.

Unless stated otherwise, the following results all hold in dimension $d = 2$, with damping/control zone equal to a full collar neighborhood of the boundary. In [ET15], the authors studied uniform stabilization and control properties of nonlinear plates with a variety of boundary conditions, notably including models with strictly dissipative polynomial nonlinearity and the Berger plate model. We also mention [CT09] where local control for Berger plates was obtained under the assumption that the linear system is controllable. Studying models such as (1.1) might open the path toward treating more challenging models such as the Von Kármán plate equation, whose nonlinearity is $[u, \Phi(u, u)]$ where Φ is the Airy stress function. Regarding the long-time dynamics and existence of compact attractors for Von Kármán and Berger plates with a variety of boundary conditions, we refer to Geredeli, Lasiecka and Webster [GLW13] and Geredeli and Webster [GW13], respectively. In these works, a nonlinear damping is considered and is emphasized that the unique continuation for such nonlinear models is a challenging problem. For a thorough study of long-time dynamics of Von Kármán models, we refer to the monograph of Chueshov and Lasiecka [CL10].

Regarding control properties, instead assuming the more general Geometric Control Condition and keeping hinged boundary conditions, in [TBE24] the local exact controllability is proved around analytic equilibria for the Von Kármán plate equation. A similar result on local control around equilibrium for the Berger equation was also obtained. We point out that, although the previous results might directly share the same framework, by employing the aforementioned control strategy it might be possible to combine the long-time dynamics results with the local control around equilibrium to obtain semiglobal control of these models. In other words, an extension of Theorem 1.4 might lead to a semiglobal controllability result within the geometric framework of the present article for such models.

In the previous paragraphs, by 'variety of boundary conditions' we mean the three main boundary conditions encountered in the literature for plate equations: hinged, clamped and free. Although

some of the previous results addressed these types of boundary conditions, it is quite challenging to obtain general results encompassing all of them. For instance, trying to translate our results to the clamped case is already an interesting problem, since, not only is the linear observability not well understood, but to our knowledge, there is no general unique continuation result available for the linear clamped plate. However, although we heavily rely on the Schrödinger decomposition of the plate operator, we believe that our analysis and the abstract propagation result of [LL24] might shed light on the study of the more involved boundary conditions. In this direction, we also refer to the work of Le Rousseau and Zongo [LRZ23] regarding the stabilization of the damped linear plate equation with boundary conditions satisfying the Lopatinskiĭ-Šapiro condition. There, by means of microlocal analysis and resolvent estimates, the authors proved that, under no geometric assumption on the damping, the decay rate is at most logarithmic. To our knowledge, there is no result for nonlinear plates under such general boundary conditions.

1.2. **Outline of the article.** Section 2 introduce some necessary preliminaries about the plate equation. Section 3 we prove Theorem 1.5 of unique continuation. Section 4 is devoted to the study of the damped plate equation (1.2), where we prove Theorem 1.2 and Theorem 1.4. Section 5 is devoted to the controllability result and contains the proof of Theorem 1.3.

1.3. **Acknowledgments.** I would like to thank Camille Laurent for drawing my attention to this problem, as well as for the helpful discussions, encouragement, and patient guidance.

2. Preliminaries

2.1. **Well-posedness.** Let $A_0 = -\Delta_g$ be the Laplace-Beltrami operator, equipped with Dirichlet boundary conditions if $\partial \mathcal{M} \neq \emptyset$. Recall that $A_0 : D(A_0) \to L^2(\mathcal{M})$ is a self-adjoint and nonnegative operator. In this case, its domain is given by $D(A_0) = H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$ on $L^2(\mathcal{M})$. Set $X = D(A_0) \times L^2(\mathcal{M})$ and introduce the densely defined operator $\boldsymbol{A} : D(\boldsymbol{A}) \to X$ given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & I \\ -A_0^2 - \beta I & 0 \end{pmatrix} \quad \text{with} \quad D(\boldsymbol{A}) = D(A_0^2) \times D(A_0).$$

For $s \geq 0$, we introduce the scale of Sobolev spaces $H_D^s = D((-\Delta_g)^{s/2})$ equipped with the inner product

$$\langle \phi, \psi \rangle_{H_D^s} = \langle (1 + A_0)^{s/2} \phi, (1 + A_0)^{s/2} \psi \rangle_{L^2(\mathcal{M})},$$

see [Gri67]. Then, for $\sigma \in [0, 1]$, X^σ denotes the space

$$X^\sigma = D(A_0^{1+\sigma}) \times D(A_0^\sigma) = H_D^{2+2\sigma} \times H_D^{2\sigma}, \tag{2.1}$$

equipped with the natural inner product. By Stone's theorem, $\boldsymbol{A}$ generates a unitary C_0-group on X and $D(\boldsymbol{A})$. By linear interpolation, so it does on X^σ for any $\sigma \in [0, 1]$.

We set $F \in C^0(X)$ to be the map

$$F : (u, v) \in X \longmapsto (0, -f(\cdot, u)) \in X.$$

If $U_0 = (u_0, u_1) \in X$ and $U = (u, \partial_t u)$, we can recast our equation (1.1) as

$$\begin{cases} \partial_t U = \boldsymbol{A} U + F(U), & t > 0, \\ U(0) = U_0, \end{cases}$$

We first state the global well-posedness for solutions of (1.1) under the asymptotic defocusing condition (1.6).

Proposition 2.1. *Let $f \in C^4(\mathbb{R} \times \mathcal{M}, \mathbb{R})$ satisfy (1.6). For any $T > 0$, any $U_0 = (u_0, u_1) \in X = H_D^2 \times L^2(\mathcal{M})$ and $h \in L^1([0, T], L^2(\mathcal{M}))$ there exists a unique solution $U(t) = (u(t), \partial_t u(t)) \in C^0([0, T], X)$ of*

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = h & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}), \\ u_{|\partial \mathcal{M}} = \Delta u_{|\partial \mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial \mathcal{M}, \\ (u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M}, \end{cases} \tag{2.2}$$

Moreover the flow map

$$\begin{cases} X \times L^1([0, T], L^2(\mathcal{M})) & \longrightarrow & C^0([0, T], X) \\ (U_0, h) & \longmapsto & U \end{cases}$$

is Lipschitz on bounded sets.

Proof. Since H_D^2 is an algebra for $d \le 3$, for any u, v with $\|u\|_{H_D^2}$, $\|u\|_{H_D^2} \le R$, by classical Sobolev embedding,

$$\|f(\cdot, u) - f(\cdot, v)\|_{L^2(\mathcal{M})} \le \left\| \int_0^1 f_s'(\cdot, v + \tau(u - v))(u - v) d\tau \right\|_{L^2(\mathcal{M})} \le C(R) \|u - v\|_{H_D^2}.$$

Therefore, F is Lipschitz on the bounded sets of X and thus the local well-posedness can be established by classical Picard iteration, see for instance [Paz83, Theorem 1.4].

To globalize the solution we need to obtain some suitable bounds on the energy. First, given that $\mathcal{M}$ is compact and assumption (1.6) forces that $s \mapsto V(\cdot, s)$ is non-increasing on $(-\infty, -R)$ and non-decreasing in $(R, +\infty)$, it follows that V attains a global infimum on $\mathcal{M} \times \mathbb{R}$, that is, $\inf_{(x,s) \in \mathcal{M} \times \mathbb{R}} V(x, s) \ge -C_0$ for some $C_0 \in \mathbb{R}$. Therefore, for any $U \in X$ it holds

$$E(U) \ge \frac{1}{2} \|U\|_X^2 - \mathrm{meas}(\mathcal{M}) C_0. \tag{2.3}$$

Second, we note that for $s \le t$, the energy E satisfies

$$E(U(t)) - E(U(s)) = \int_s^t \int_{\mathcal{M}} h \partial_t u \, dx \, d\tau,$$

from which follows

$$E(U(t)) \le E(U(s)) + C \|h\|_{L^1([s,t], L^2(\mathcal{M}))} \sup_{\tau \in [s,t]} \left(E(U(\tau)) + C_1 \right)^{1/2}, \tag{2.4}$$

with $C_1 := \mathrm{meas}(\mathcal{M}) C_0$. By Gronwall's inequality, we readily get that the energy remains bounded in bounded intervals. So, both estimates (2.3) and (2.4) give us a bound on the norm for the solution to our equation. Given that f is smooth and satisfies $f(x, 0) = 0$ for every $x \in \partial \mathcal{M}$, Theorem A.2 give us $\|F(U)\|_X \le C(\|u\|_{L^\infty}) \|U\|_X$. So, if U belongs to a bounded set in X, the Sobolev embedding $H_D^2 \hookrightarrow L^\infty$ tells us that the energy remains bounded and thus does not blow up in finite time.

Finally, the continuity estimate follows by using Duhamel's formula along with the fact that the nonlinearity is Lipschitz on bounded sets and a Gronwall type argument. $\square$

Remark 2.2. Note that the condition $\Delta u_{|\partial \mathcal{M}} = 0$ which does not make sense at this level of regularity is meant as an extension of the related semigroup. The nonlinear equation (2.2) is meant in the sense of the Duhamel formula.

Let us introduce the linear bounded operator

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma^2(x) \end{pmatrix}$$

so that the damped system (1.2) can be rewritten as

$$\begin{cases} \partial_t U = (\boldsymbol{A} + \mathbf{B})U + F(U), & t > 0, \\ \qquad U(0) = U_0, \end{cases}$$

With minor modifications, we can establish a similar result for the damped system (1.2).

Proposition 2.3. *Let f satisfy (1.6). For any $U_0 = (u_0, u_1) \in X = H_D^2 \times L^2(\mathcal{M})$ there exists a unique solution $U(t) = (u(t), \partial_t u(t)) \in C^0(\mathbb{R}, X)$ of the damped plate equation (1.2). Moreover, its energy $E(u, \partial_t u)$ is non-increasing in time and for any $T > 0$, the flow map $U_0 \in X \longmapsto U \in C^0([-T, T], X)$ is Lipschitz on bounded sets.*

Proof. The only modification on the sketch of the previous proof is that now the energy is dissipative and it satisfies

$$\frac{d}{dt} E(U) = - \int_{\mathcal{M}} \gamma^2(x) |\partial_t u|^2 dx.$$

The remaining details are filled in a similar fashion. $\qquad\square$

2.2. **Nonlinearity properties.** In what follows, we denote by $\mathbb{B}_R(Y)$ the ball of radius $R > 0$ centered at 0 of some Banach space Y. We now show that the nonlinearity is more regular than it seems to be as a map defined on the energy space X.

Proposition 2.4. *Let $f \in C^4(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ with $f(x, 0) = 0$ for $x \in \partial\mathcal{M}$. For every $\sigma \in [0, 1)$, F maps any bounded set $\mathcal{B}$ of $X = H_D^2 \times L^2$ into a bounded set $F(\mathcal{B})$ contained in $X^\sigma = (H^{2+2\sigma} \cap H_D^2) \times H_D^{2\sigma}$. Furthermore, $F(\mathcal{B})$ is relatively compact in X^σ. Moreover, for any $U, V \in \mathcal{B}$ then*

$$\|F(U) - F(V)\|_{X^\sigma} \leq C(\mathcal{B}) \|U - V\|_X. \tag{2.5}$$

Proof. Since $d \leq 3$ and $f \in C^4(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ with $f(x, 0) = 0$ for $x \in \mathcal{M}$, Theorem A.1 shows that $f : H_D^2 \to H_D^2$ is well-defined and $\|f(u)\|_{H_D^2} \leq C(\|u\|_{L^\infty(\mathcal{M})}) \|u\|_{H_D^2}$. Thus, for any $\sigma \in [0, 1)$, since the compact embedding $H_D^2 \hookrightarrow H_D^{2\sigma}$ holds by Rellich-Kondrachov's theorem, we get that the set $\{f(u) \mid (u, v) \in \mathcal{B}\}$ is relatively compact in $H_D^{2\sigma}$. Thus, due to the structure of the nonlinearity, we readily see that F maps bounded sets of X into relatively compact sets of X^σ.

Let us write

$$f(\cdot, u) - f(\cdot, v) = (u - v) \int_0^1 \left(f_s'(\cdot, v + \tau(u - v)) - f_s'(\cdot, 0) \right) d\tau + (u - v) f_s'(\cdot, 0).$$

Since $f_s' \in C^3(\mathcal{M} \times \mathbb{R}, \mathbb{R})$, by Theorem A.1 we readily get $f_s' - f_s'(\cdot, 0) : H_D^\nu \to H_D^\nu$ with $\nu \in (3/2, 2)$ is well-defined and $\|f_s'(\cdot, w) - f_s'(\cdot, 0)\|_{H_D^\nu} \leq C(\|w\|_{L^\infty}) \|w\|_{H_D^\nu}$. In particular, using that H_D^2 is an algebra and Sobolev embeddings, for any $u, v \in H_D^2$ we have

$$\|f(u) - f(v)\|_{H_D^{2\sigma}} \leq C \|u - v\|_{H_D^2} \left(1 + \|u\|_{H_D^2} + \|v\|_{H_D^2} \right),$$

where the constant C depends only on f and on the L^∞ norm of both u and v. By Sobolev embedding, this estimate directly implies (2.5), since, for any $U, V \in \mathcal{B}$,

$$\|F(U) - F(V)\|_{X^\sigma} = \|f(u) - f(v)\|_{H_D^{2\sigma}} \leq C \|u - v\|_{H_D^2} \leq C \|U - V\|_X,$$

where C only depends on f and $\mathcal{B}$. $\qquad\square$

2.3. **Linear observability.** Now we state a suitable observability inequality for the linear plate equation, which will be employed later on. Obtaining such observability inequality follows from Lebeau's strategy [Leb92] and the observation that, under hinged boundary conditions, the bi-Laplace operator is precisely the square root of the Dirichlet-Laplace operator. Even though Lebeau's was first concerned with obtaining observability of linear plates under the GCC, as long as observability for Schrödinger holds true, this abstract strategy allows to transfer observability results known from the linear Schrödinger equation, to the linear hinged plate equation. This explains the geometric assumption on the pair $(\mathcal{M}, \omega)$ and it notably simplifies the arguments. The following result is borrowed from [LL24, Section 4].

Proposition 2.5. [LL24, Proposition 4.6] *Let $T_0 > 0$ and ω satisfying Assumption 1 on $(0, T_0)$. Then, for any $0 < T_0 < T < +\infty$ and any $b_\omega \in C^\infty(\mathcal{M})$ so that $b_\omega = 1$ on ω, there exists $C > 0$ such that for any $s \in [0, 2]$, any $(z_0, z_1) \in H_D^2 \times L^2(\mathcal{M})$ and associated solution z of*

$$\begin{cases} \partial_t^2 z + \Delta_g^2 z = 0 & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}), \\ z_{|\partial\mathcal{M}} = \Delta_g z_{|\partial\mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial\mathcal{M}, \\ (z, \partial_t z)(0) = (z_0, z_1) & x \in \mathcal{M}, \end{cases}$$

we have, if $\partial M \neq \emptyset$

$$\|(z_0, z_1)\|_{H_D^{2+s} \times H_D^s(\mathcal{M})}^2 \leq C \int_0^T \|b_\omega \Delta_g z(t)\|_{H_D^s(\mathcal{M})}^2 dt, \tag{2.6}$$

and if $\partial M = \emptyset$

$$\|(z_0, z_1)\|_{H^{2+s} \times H^s(\mathcal{M})}^2 \leq C \int_0^T \|b_\omega z(t)\|_{H^{2+s}(\mathcal{M})}^2 dt. \tag{2.7}$$

The following observability inequality with time-independent potential will be used later on.

Corollary 2.6. *Let $V \in C^\infty(\mathcal{M})$ and ω satisfying Assumption 1 in time $T_0 > 0$. Let $\gamma \in C^\infty(\mathcal{M})$ be as in Section 1.1.1. Under the hypotheses of Theorem 2.5, for any $T > T_0$ there exists $C > 0$ such that, for any $(z_0, z_1) \in H_D^2 \times L^2(\mathcal{M})$ and associated solution z of*

$$\begin{cases} \partial_t^2 z + \Delta_g^2 z + \beta z + V(x)z = 0 & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}), \\ z_{|\partial\mathcal{M}} = \Delta z_{|\partial\mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial\mathcal{M}, \\ (z, \partial_t z)(0) = (z_0, z_1) & x \in \mathcal{M}, \end{cases} \tag{2.8}$$

we have

$$\|(z_0, z_1)\|_{H_D^2 \times L^2(\mathcal{M})}^2 \leq C \int_0^T \|\gamma z(t)\|_{H_D^2(\mathcal{M})}^2 dt. \tag{2.9}$$

Proof. Given that the addition of the potential is a compact perturbation, we can deduce the observability estimate (2.9) from the free observability estimate (2.7), which holds whenever $\partial\mathcal{M}$ is empty or not. We briefly sketch the proof, based on a slight variation of the abstract result [DO18, Corollary 1.3, Remark 1.5], where exact controllability can be replaced by observability, according to the duality framework given by the HUM method (see Section 5.1.1 below). Indeed, they provide the necessary arguments to remove the compact term in the proof of the observability inequality in the already classical compactness-uniqueness argument under the presence of a compact perturbation. They rely on the observability of the linear system and a unique continuation result for eigenfunctions, which they refer to as Fattorini-Hautus test.

In what follows we check the required hypothesis. If $Z := (z, \partial_t z) \in C^0([0, T], X)$ then (2.8) can be written as a first order system

$$\begin{cases} \partial_t Z = (\boldsymbol{A} + \boldsymbol{V})Z, & t \in (0, T) \\ Z(0) = Z_0, \end{cases}$$

where $\boldsymbol{V}$ is given by

$$\boldsymbol{V} := \begin{pmatrix} 0 & 0 \\ -V(x) & 0 \end{pmatrix}.$$

First, by Theorem 2.5 observability holds for any $T > T_0$ for $\boldsymbol{A}$ with observation operator $\mathbf{C}(\psi_1, \psi_2) = \gamma(x)\psi_1$. Second, note that $\boldsymbol{A_V} := \boldsymbol{A} + \boldsymbol{V}$ satisfies $D(\boldsymbol{A_V}) = D(\boldsymbol{A})$ and that $\boldsymbol{V} \in \mathcal{L}(X)$ is a compact operator. Then, it only remains to check the unique continuation property for eigenfunctions of $\boldsymbol{A_V}$: if $\boldsymbol{A_V}(v_\lambda, w_\lambda) = \lambda(v_\lambda, w_\lambda)$ and $\mathbf{C}(v_\lambda, w_\lambda) = 0$ with $(v_\lambda, w_\lambda) \in X$ and $\lambda \in \mathbb{C}$, then $(v_\lambda, w_\lambda) = (0, 0)$. On the one hand, the condition $\boldsymbol{A_V}(v_\lambda, w_\lambda) = \lambda(v_\lambda, w_\lambda)$ leads to

$$\begin{cases} -(\Delta_g^2 + \beta + V(x))v_\lambda = \lambda^2 v_\lambda, & x \in \mathcal{M}, \\ v_\lambda = 0, & x \in \omega. \end{cases}$$

By elliptic regularity and bootstrap, we can show that $v_\lambda \in H^3 \cap H_D^1(\mathcal{M})$. In particular, we can apply the unique continuation for (evolution) plate operators Theorem 3.1 below to show that $v_\lambda \equiv 0$. On the other hand, for $\lambda \neq 0$, we have $w_\lambda = \lambda v_\lambda = 0$, and for $\lambda = 0$ we directly obtain $w_\lambda = 0$. This proves that linear observability holds in time $T > T_0$ for $\boldsymbol{A_V}$ with observation operator $\mathbf{C}$, that is, estimate (2.9) holds. $\square$

Remark 2.7. Regarding the unique continuation for eigenfunctions of the bi-Laplace operator, we refer to Le Rousseau and Robbiano [LRR20] and the references therein, where more complicated situations are addressed.

2.4. **About the stabilization strategy.** The following well-known result gives us a criterion to establish the nonlinear energy decay for defocusing solutions of (1.2) as stated in the introduction.

Lemma 2.8. *Let f satisfy the defocusing assumption* (1.3). *The property (ED) holds if and only if there exists $T > 0$ such that for every E_0, there exists a constant $C > 0$ such that*

$$E(U(0)) \leq C\big(E(U(0)) - E(U(T))\big) = C \iint_{[0,T] \times \mathcal{M}} \gamma^2(x)|\partial_t u(t, x)|^2 dxdt \qquad (2.10)$$

for all solutions u of (1.2) *with $E(U(0)) \leq E_0$.*

Proof. If (ED) holds, then for $T > 0$ large enough so that $1 - Ce^{-\lambda T} > 0$,

$$E(U(0)) - E(U(T)) \geq \big(1 - Ce^{-\lambda T}\big)E(U(0)).$$

If (2.10) holds, using $E(U(T)) \leq E(U(0))$, we get directly that

$$E(U(T)) \leq \frac{C}{C+1}E(U(0)).$$

Consequently,

$$E(U(kT)) \leq \frac{C}{C+1}E(U((k-1)T)) \leq \left(\frac{C}{C+1}\right)^2 E(U((k-2)T)) \leq \ldots \leq C_1^k E(U(0)),$$

with $C_1 < 1$. If $t \in [kT, (k+1)T)$, observing that $k = \lfloor \frac{t}{T} \rfloor > \frac{t}{T} - 1$ and $\ln \frac{1}{C_1} > 0$, we get

$$C_1^k = \exp(k \ln C_1) = \exp(-k \ln \tfrac{1}{C_1}) \leq \exp\big(-\ln \tfrac{1}{C_1} - \tfrac{t}{T} \ln \tfrac{1}{C_1}\big) = \frac{1}{C_1} \exp\big(-\frac{\ln \frac{1}{C_1}}{T} t\big).$$

Consequently, for some constants $C > 0$ and $\lambda > 0$, we get

$$E(U(t)) \leq E(U(kT)) \leq Ce^{-\lambda t} E(U(0)).$$

$\square$

This link in between exponential stabilization and observability inequalities was first introduced by Haraux [Har89] for second order linear systems. In view of this strategy and following the same strategy of proof of Theorem 2.6, we get the observability inequality that leads to the decay of the linear plate with localized damping.

Proposition 2.9. *If ω satisfies Assumption 1, then there exists two positive constants $C > 0$ and $\lambda > 0$ such that for any $\sigma \in [0,1]$ and $t \geq 0$,*

$$\|e^{t(\boldsymbol{A}+\boldsymbol{B})}\|_{\mathcal{L}(X^\sigma)} \leq Ce^{-\lambda t}.$$

Remark 2.10. The full range of exponents $\sigma \in [0,1]$ is obtained by using suitable energy estimates to establish the case $\sigma = 1$ and then linear interpolation.

3. Unique continuation and equilibriums

As stated in the introduction, a crucial property on which the technique relies is a unique continuation property for the nonlinear plate equation (1.1). It is built upon two blocks. The first one is a recent result from [FLL25], stating a unique continuation property for the linear plate equation with time-dependent lower order terms. Actually, this result is stated as a local unique continuation property. However, as pointed out in by the authors, such result can be globalized by propagation through a suitable family of hypersurfaces, whose construction can be found in [LL19, Proof of Theorem 6.7]. The global version is stated below.

Theorem 3.1. [FLL25, Section 1.3.5] *Let $T > 0$ and $\mathcal{M} = \operatorname{Int}\mathcal{M} \sqcup \partial\mathcal{M}$ be a connected smooth manifold with or without boundary $\partial\mathcal{M}$. Suppose that $g \in W^{3,\infty}_{\mathrm{loc}}(\operatorname{Int}(\mathcal{M}))$ is a Riemannian metric on $\operatorname{Int}\mathcal{M}$, that $\mathsf{q} \in \mathcal{G}^2((0,T); L^\infty_{\mathrm{loc}}(\operatorname{Int}\mathcal{M}; \mathbb{C}))$, that b is a one form with all components belonging to $\mathcal{G}^2((0,T); L^\infty_{\mathrm{loc}}(\operatorname{Int}\mathcal{M}; \mathbb{C}))$, and consider the differential operator*

$$\mathcal{T}_{\mathsf{b},\mathsf{q}} := \partial_t^2 + \Delta_g^2 + \mathsf{b}\cdot\nabla_g + \mathsf{q}(t,x),$$

where Δ_g is the Laplace-Beltrami operator on $\operatorname{Int}\mathcal{M}$, ∇_g the Riemannian gradient.
   Then given ω a nonempty open set of $\mathcal{M}$, we have

$$\begin{cases} \mathcal{T}_{\mathsf{b},\mathsf{q}}u = 0 \ in \ (0,T)\times\operatorname{Int}\mathcal{M} \\ u \in H^1_{\mathrm{loc}}(0,T; H^3_{\mathrm{loc}}(\operatorname{Int}\mathcal{M})) \qquad \Longrightarrow \ u = 0 \ in \ (0,T)\times\operatorname{Int}\mathcal{M}. \\ u = 0 \ in \ (0,T)\times\omega \end{cases}$$

Remark 3.2. For a Banach space Y, from the definition of s-Gevrey spaces it follows that any analytic function $f : (0,T) \to Y$ also belongs to $\mathcal{G}^s((0,T), Y)$ for $s > 1$, see [FLL25, Definition 1.1].

The second block is a propagation of regularity result, or to be more precise, propagation of analyticity in time for solutions which are observed to be analytic in some region from where linear observability holds.

Theorem 3.3. [LL24, Theorem 1.6] *Let $(\mathcal{M}, g)$ be a compact connected manifold with (or without) boundary of dimension $d \leq 3$. Let $\widetilde{\omega} \subset \mathcal{M}$ be a nonempty open set and suppose that there exists $\omega \Subset \widetilde{\omega}$ satisfying Assumption 1 for some $T_0 > 0$. Assume that $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is smooth and real analytic on the second variable. If $T > T_0$, let $(u, \partial_t u) \in C^0([0,T], H^2_D \times L^2)$ be a solution of*

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \beta u + f(x,u) = 0 & (t,x)\in[0,T]\times\operatorname{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & (t,x)\in[0,T]\times\partial\mathcal{M}, \end{cases} \tag{3.1}$$

so that for any cutoff function $\chi \in C_c^\infty(\mathcal{M})$ whose support is contained in $\widetilde{\omega}$, then $t \in (0,T) \mapsto \chi u(t,\cdot) \in H^{3+\varepsilon}(\mathcal{M}) \cap H^1_D(\mathcal{M})$ is analytic for one $\varepsilon > 0$. Then $t \in (0,T) \mapsto (u(t,\cdot), \partial_t u(t,\cdot)) \in H^4 \cap H^1_D \times H^2 \cap H^1_D$ is analytic.

Remark 3.4. Observe that in [LL24] this result is not written for f with an extra x-dependency. However, as long as f is smooth with respect to x, the adaptation is straightforward as we only need to ensure that [LL24, Assumption 3] is verified. The latter follows directly by using the nonlinear estimates Theorem A.1 as indicated in [LL24, Proposition 3.2, Proposition 4.1].

This result can be viewed as a finite-time adaptation of abstract results by Hale and Raugel [HR03] in the concerning the regularity of compact attractors. It is likely that analyticity in infinite time holds true after Hale and Raugel's results. However, until now Theorem 1.5 seemed to be hard to obtain because, for instance, no unique continuation result for linear plates was available under non-restrictive geometric assumptions. In this direction, the recent result (3.1) of [FLL25] underpins the importance of our result about propagation of analyticity in finite time. This property is the crucial not only because it enables us to use unique continuation for linear plates Theorem 3.1, but also for deriving the corresponding observability inequality for the nonlinear equation.

With these two results at hand, we are in position to prove the unique continuation property stated in the introduction.

Proof of Theorem 1.5. Let $U(t) = (u(t), \partial_t u(t))$ be a solution of (1.1) in $C^0([0,T], X^\sigma)$. By assumption $\omega \Subset \widetilde{\omega}$ and ω satisfies Assumption 1. Let $\chi \in C_c^\infty(\mathcal{M})$ be a smooth compactly supported function in $\widetilde{\omega}$ such that $\chi = 1$ on ω and $\partial_{\vec{n}}\chi = 0$ on $\partial\mathcal{M}$. By hypothesis, in $[0,T] \times \widetilde{\omega}$ we have $\partial_t u = 0$ and thus

$$\Delta_g^2(\chi u) + \beta(\chi u) + \chi f(x, u) = [\Delta_g^2, \chi]u.$$

By elliptic regularity and bootstrap (see, for instance [GT01, Proposition 9.19]), we readily get that, for some small $\varepsilon > 0$, the application $t \in (0,T) \mapsto \chi u(t, \cdot) \in H^{3+\varepsilon}(\mathcal{M}) \cap H_D^1(\mathcal{M})$ is well-defined and analytic, as it does not depend on t. Thus Theorem 3.3 applies and we get that $t \in (0,T) \mapsto U(t) \in H^4 \cap H_D^1 \times H_D^2$ is analytic.

Since u is smooth with respect to t and f is smooth in both variables, by writing

$$(\Delta_g^2 + \beta)u = -\partial_t^2 u - f(x, u)$$

we get that $\Delta_g^2 u \in L^2(\mathcal{M})$ and so $u \in H^4(\mathcal{M})$. We differentiate the above equation to obtain

$$(\Delta_g^2 + \beta)^2 u = (\Delta_g^2 + \beta)(-\partial_t^2 u - f(x, u)) = -\partial_t^2(\Delta_g^2 + \beta)u - (\Delta_g^2 + \beta)f(x, u)$$
$$= \partial_t^4 u + \partial_s f(x, u)\partial_t^2 u + \partial_s^2 f(x, u)(\partial_t^2 u)^2 - (\Delta_g^2 + \beta)f(x, u)$$

which shows that u belongs to $H^8(\mathcal{M})$. This process can be repeated as many times as wanted, so by classical Sobolev embedding we get that $u = u(t, x)$ is smooth with respect to x. In particular, $(t, x) \in (0,T) \times \text{Int}(\mathcal{M}) \mapsto u(t, x)$ belongs to $C^\infty\big((0,T) \times \text{Int}(\mathcal{M})\big)$.

Set $z = \partial_t u$ and observe that z solves

$$\begin{cases} \partial_t^2 z + \Delta_g^2 z + \beta z + f_s'(x, u)z = 0 & (t, x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ z_{\partial\mathcal{M}} = \Delta z_{|\partial\mathcal{M}} = 0 & (t, x) \in [0,T] \times \partial\mathcal{M}, \\ z = 0 & (t, x) \in [0,T] \times \omega. \end{cases} \tag{3.2}$$

By the previous discussion $\mathsf{q} : (t, x) \mapsto f'(x, u(t, x))$ is bounded, is not only smooth in x but also analytic in t and in particular, Gevrey 2 in time, see Theorem 3.2. As u is smooth in both variables, so is z and we can apply Theorem 3.1 to get that $z \equiv 0$ everywhere. This in turn means that $u(t, x) = u(x)$ is constant in time and henceforth it solves

$$\begin{cases} \Delta_g^2 u + \beta u + f(x, u) = 0 & x \in \text{Int}(\mathcal{M}), \\ u = \Delta u = 0 & x \in \partial\mathcal{M}. \end{cases} \tag{3.3}$$

Moreover, multiplying the latter equation by u and integrating by parts leads us to the identity

$$0 \le \int_{\mathcal{M}} \big(|\Delta_g u(x)|^2 + \beta|u(x)|^2\big)dx = -\int_{\mathcal{M}} u(x)f(x, u(x))dx.$$

Under the assumption that $sf(x, s) \geq 0$ for all $(x, s) \in \mathcal{M} \times \mathbb{R}$, the above identity, the connectedness of $\mathcal{M}$ and the boundary condition imply that $u \equiv 0$ everywhere. In the case $\partial \mathcal{M} = \emptyset$, we get

$$0 \leq \int_{\mathcal{M}} \left(|\Delta_g u(x)|^2 + \beta |u(x)|^2\right) dx = -\int_{\mathcal{M}} u(x) f(x, u(x)) dx \leq -\alpha \int_{\mathcal{M}} |u(x)|^2 dx,$$

which directly implies $u \equiv 0$ everywhere. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. STABILIZATION AND LONG TIME DYNAMICS

4.1. **Observability inequality.** By Theorem 2.8, the stabilization result Theorem 1.2 follows directly once the corresponding observability inequality is established, which is the content of the following result. We employ an strategy introduced by Dehman, Lebeau and Zuazua [DLZ03], where microlocal propagation tools and unique continuation are used in a key fashion to stabilize the subcritical wave equation in $\mathbb{R}^3$. Here, following a modification of these ideas for the wave equation introduced by Laurent and the author [LL24, Section 3.5], we use the linear observability inequality and the compactness properties of the nonlinearity to perform the usual propagation of compactness argument. We point out that, for the subcritical wave equation, Joly and Laurent [JL13] introduced a similar modification based on the decay of the linear semigroup, which led them to rely on a unique continuation property in infinite time. In our case, if we run such argument we might be able to obtain the observability inequality for some $T > 0$. However, here we obtain the observability for any $T > T_0$ given that the Schrödinger equation is L^2−observable in time $T_0 > 0$. This further highlights the importance of the unique continuation property in finite time Theorem 3.3.

Theorem 4.1. *Assume ω satisfies Assumption 1 for some $T_0 > 0$. Assume that f is analytic in the second variable and defocusing, that is, satisfying* (1.3). *Then, for any $T > T_0$ and $E_0 > 0$, there exists $C > 0$ so that for any $(u_0, u_1) \in H_D^2 \times L^2(\mathcal{M})$, with $\|(u_0, u_1)\|_{H_D^2 \times L^2} \leq E_0$, the unique solution of* (1.2) *satisfies*

$$E(U(0)) \leq C \iint_{[0,T] \times \mathcal{M}} \gamma^2(x) |\partial_t u(t, x)|^2 dx dt. \tag{4.1}$$

Proof. We argue by contradiction. Let $(u_{0,n}, u_{1,n}) \in H_D^2 \times L^2(\mathcal{M})$ be a sequence of initial data with $\|(u_{0,n}, u_{1,n})\|_{H_D^2 \times L^2} \leq E_0$, so that the unique solution $(u_n, \partial_t u_n) \in C^0([0, T], X)$ to (1.1) satisfies

$$\iint_{[0,T] \times \mathcal{M}} \gamma^2(x) |\partial_t u_n(t, x)|^2 dx dt \leq \frac{1}{n} E(U_0^n) \leq \frac{1}{n} E_0. \tag{4.2}$$

Up to taking a subsequence, we can assume that $(u_{0,n}, u_{1,n})$ converges weakly to $(u_0, u_1) \in H_D^2 \times L^2$. The global well-posedness theory for (1.2) and the defocusing assumption (1.3) implies that (u_n) is globally bounded in $C^0([0, T], H_D^2)$, see Theorem 2.1. In particular, up to a subsequence, we can assume that u_n converges weakly to some $u \in L^2([0, T], H_D^2)$ for this topology.

Step 1. Identification of the weak limit. The first step is to prove that $U = (u, \partial_t u)$ is solution of the undamped nonlinear equation with initial data $U_0 = (u_0, u_1) \in H_D^2 \times L^2$. The well-posedness theory allows us to consider the following nonlinear solution with initial data U_0,

$$V(t) = e^{t(\boldsymbol{A}+\boldsymbol{B})} U_0 + \int_0^t e^{(t-s)(\boldsymbol{A}+\boldsymbol{B})} F(V(s)) ds = V_{\text{lin}} + V_{\text{Nlin}},$$

and we want to prove that $U = V$.

Since the sequence $(u_n, \partial_t u_n)$ is bounded in $C^0([0, T], H_D^2 \times L^2)$, as an application of Aubin-Lions Lemma (see [Sim87, Corollary 4]), we obtain that, for any $0 < \eta < 1$, still up to a subsequence which we denote the same, u_n converges strongly to u in $C^0([0, T], H^{2-\eta}(\mathcal{M}))$. By leveraging that $d \leq 3$ and thus we are working on an algebra, by Sobolev embedding if we take $0 < \eta < 1/2$ we get

that u_n converges strongly to u in $L^\infty([0,T] \times \mathcal{M})$. Moreover, by estimates similar to the ones in Theorem 2.4, we obtain

$$\|f(\cdot, u_n) - f(\cdot, u)\|_{L^1([0,T], L^2(\mathcal{M}))} \leq \left\| \int_0^1 f_s'(\cdot, u_n + \tau(u_n - u))(u_n - u)d\tau \right\|_{L^1([0,T], L^2(\mathcal{M}))}$$

$$\leq C\|u_n - u\|_{L^\infty([0,T] \times \mathcal{M})}$$

where $C > 0$ is a constant that depends on f and on E_0. The previous estimate implies that $f(u_n) \to f(u)$ strongly in $L^1([0,T], L^2(\mathcal{M}))$. From the Duhamel's formulation

$$U^n(t) = e^{t(\mathbf{A}+\mathbf{B})}U_0^n + \int_0^t e^{(t-s)(\mathbf{A}+\mathbf{B})}F(U^n(s))ds = U_{\text{lin}}^n(t) + U_{\text{Nlin}}^n(t), \tag{4.3}$$

with

$$\|U_{\text{Nlin}}^n - V_{\text{Nlin}}\|_{C^0([0,T], X)} \xrightarrow[n\to\infty]{} 0. \tag{4.4}$$

Since the map $U_0 \mapsto e^{\cdot(\mathbf{A}+\mathbf{B})}U_0$ is linear continuous from X into $L^2([0,T], X)$, it is continuous for the corresponding weak topology as well. Therefore, since U_0^n converges weakly to U_0 in X, $e^{\cdot(\mathbf{A}+\mathbf{B})}U_0^n$ converges weakly to $e^{\cdot(\mathbf{A}+\mathbf{B})}U_0 = V_{\text{lin}}$ in $L^2([0,T], X)$. It follows that U^n converges weakly to V and consequently $U = V$.

We obtained that $U = (u, \partial_t u) \in C^0([0,T], H_D^2 \times L^2(\mathcal{M}))$ is a solution of

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = 0 & (t, x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \Delta u_{|\partial\mathcal{M}} = 0 & (t, x) \in [0,T] \times \partial\mathcal{M}, \\ (u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M}. \end{cases}$$

Let $\widetilde{\omega}$ be the set of points such that $\gamma(x) > \gamma_0/2 > 0$, which satisfies $\omega \Subset \widetilde{\omega}$ given that γ is smooth and $\mathcal{M}$ is compact. Indeed, the set $\omega_0 := \{x \in \mathcal{M} \mid \gamma(x) \geq \frac{2}{3}\gamma_0 > 0\}$ is closed (hence compact) and satisfies $\omega \subset \omega_0 \subset \widetilde{\omega}$. Using (4.2) and taking weak limit in the equation satisfied by $(u_n, \partial_t u_n)$, by looking at the equation satisfied by $(u, \partial_t u)$, we readily get $\partial_t u = 0$ on $[0,T] \times \widetilde{\omega}$. As we are assuming that f is defocusing, we are then in position to apply the unique continuation property Theorem 1.5, from which we deduce that $u \equiv 0$. In particular, $V_{\text{Nlin}} = 0$ and from (4.4) we get

$$\|U_{\text{Nlin}}^n\|_{C^0([0,T], X)} \xrightarrow[n\to\infty]{} 0. \tag{4.5}$$

Using the linear observability inequality from Theorem 2.5 and Duhamel's formula (4.3), we have

$$\|U_0^n\|_X^2 \leq C \int_0^T \|\gamma \partial_t u_{\text{lin}}^n\|_{L^2(\mathcal{M})}^2 dt$$

$$\leq 2C \int_0^T \|\gamma \partial_t u^n\|_{L^2(\mathcal{M})}^2 dt + 2C \int_0^T \|\gamma \partial_t u_{\text{Nlin}}^n\|_{L^2(\mathcal{M})}^2 dt.$$

From (4.2) and (4.5) together, we obtain $\alpha_n := \|U_0^n\|_X \xrightarrow[n\to\infty]{} 0$.

Step 2. Linearization argument. Since the sequence of initial data converges strongly to 0, we can linearize to consider the nonlinearity as a small perturbation of the linear problem, for which we know the observability applies. Let us denote by $w_n = u_n/\alpha_n$, which is a mild solution of

$$\begin{cases} \partial_t^2 w_n + \Delta_g^2 w_n + \beta w_n + \alpha_n^{-1} f(x, \alpha_n w_n) = 0 & (t, x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ w_{n|\partial\mathcal{M}} = \Delta w_{n|\partial\mathcal{M}} = 0 & (t, x) \in [0,T] \times \partial\mathcal{M}, \\ (w_n, \partial_t w_n)(0) = (w_{n,0}, w_{n,1}) & x \in \mathcal{M}, \end{cases}$$

with $\|(w_{n,0}, w_{n,1})\|_X = 1$. By expanding $f(\cdot, s) = f(\cdot, 0) + f'(\cdot, 0)s + r(\cdot, s)$, the nonlinearity $f_n(s) := \alpha_n^{-1} f(\alpha_n s)$ can be written as $f_n(s) = f'(0)s + \alpha_n^{-1} r(\alpha_n s)$. By Theorem A.2 and Sobolev embedding,

$r_n(s) := \alpha_n^{-1} r(\alpha_n s)$ satisfies, uniformly in $n \in \mathbb{N}$,

$$\|r_n(w_n)\|_{L^\infty([0,T],L^2(\mathcal{M}))} \leq C\alpha_n \|w_n\|^2_{C^0([0,T],H^2_D(\mathcal{M}))}$$

Introducing

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} 0 & I \\ -A_0^2 - \beta - f'(\cdot,0) & 0 \end{pmatrix} \quad \text{and} \quad R_n \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -r_n(u) \end{pmatrix},$$

we write

$$W^n(t) = e^{t\widetilde{\boldsymbol{A}}} W_0^n + \int_0^t e^{(t-s)\widetilde{\boldsymbol{A}}} R_n(W^n(s)) ds = W_{\text{lin}}^n + W_{\text{Nlin}}^n.$$

By employing the observability inequality of Theorem 2.6 with potential $V = f'(\cdot,0)$ for the linear part and the previous estimates for the nonlinear part, we obtain

$$\|W_{\text{lin}}^n\|_{C^0([0,T],X)} \leq C,$$
$$\|W_{\text{Nlin}}^n\|_{C^0([0,T],X)} \leq C\alpha_n \|W^n\|^2_{C^0([0,T],X)},$$

with $C = C(E_0, T) > 0$. By putting together both bounds, we obtain

$$\|W^n\|_{C^0([0,T],X)} \leq C + C\alpha_n \|W^n\|^2_{C^0([0,T],X)}.$$

A bootstrap argument [BG99, Lemma 2.2] allows us to prove that, for n large enough,

$$\|W^n\|_{C^0([0,T],X)} \leq 2C,$$

which, in turn, gives us

$$\|W_{\text{Nlin}}^n\|_{C^0([0,T],X)} \leq C\alpha_n.$$

We now treat the linear part by using the observability estimate from Theorem 2.6. Thus,

$$\|(w_{0,n}, w_{1,n})\|_X^2 \leq C \int_0^T \|\gamma \partial_t w_{\text{lin}}^n\|^2_{L^2(\mathcal{M})} dt$$

$$\leq 2C \int_0^T \|\gamma \partial_t w^n\|^2_{L^2(\mathcal{M})} dt + 2C \int_0^T \|\gamma \partial_t w_{\text{Nlin}}^n\|^2_{L^2(\mathcal{M})} dt.$$

For the first term, (4.2) with the scaling $w_n = u_n/\alpha_n$ and $\alpha_n = \|(u_{0,n}, u_{1,n})\|_X$ can be written as

$$\iint_{[0,T]\times\mathcal{M}} \gamma^2(x) |\partial_t w_n(t,x)|^2 dx dt \leq \frac{1}{n}.$$

Concerning the second term, by energy estimates we get

$$\int_0^T \|\gamma \partial_t w_{\text{Nlin}}^n\|^2_{L^2(\mathcal{M})} dt \leq C \|W_{\text{Nlin}}^n\|^2_{C^0([0,T],X)} \leq C\alpha_n^2.$$

Putting them all together gives us

$$E(w_{0,n}, w_{1,n}) \leq \frac{C}{n} + C\alpha_n^2.$$

Therefore $E(w_{0,n}, w_{1,n}) \to 0$ as $n \to \infty$, which is a contradiction with the initial assumption since $1 = \|(w_{0,n}, w_{1,n})\|_X \leq CE(w_{0,n}, w_{1,n})$ for every $n \in \mathbb{N}$. $\qquad\square$

4.2. **Existence of a compact global attractor.** From now on, we will assume that f satisfies the asymptotic condition (1.6), that is,

$$f(x, 0) = 0 \text{ for every } x \in \partial \mathcal{M} \text{ and } sf(x, s) \geq 0 \text{ for every } (x, s) \in \mathcal{M} \times \{|s| \geq R\}.$$

We recall that $(e, 0) \in H_D^2 \times L^2(\mathcal{M})$ is an equilibrium point of (1.1) if it satisfies

$$\begin{cases} \Delta_g^2 e + \beta e + f(x, e) = 0 & x \in \text{Int}(\mathcal{M}), \\ e_{|\partial \mathcal{M}} = \Delta e_{|\partial \mathcal{M}} = 0 & x \in \partial \mathcal{M}. \end{cases} \tag{4.6}$$

4.2.1. *The dynamical system is asymptotically smooth.* This property follows exactly as in [JL13, Proposition 4.1]. We recall the argument for the sake of completeness.

Proposition 4.2. *Assume ω satisfies Assumption 1. Let $f \in C^4(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfying (1.6), let $(u_0^n, u_1^n)_{n \in \mathbb{N}}$ be a sequence of initial data which is bounded in X and let $(u_n)_{n \in \mathbb{N}}$ be the corresponding solutions to the damped plate equation (1.2). Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of times such that $t_n \to +\infty$ as $n \to \infty$.*

Then, there exist subsequences $(u_{n_k})_k$ and $(t_{n_k})_k$ and a global solution $U_\infty = (u_\infty, \partial_t u_\infty)$ of (1.2) defined for all $t \in \mathbb{R}$ such that

$$(u_{n_k}, \partial_t u_{n_k})(t_{n_k} + t) \xrightarrow[k \to \infty]{} (u_\infty, \partial_t u_\infty)(t) \quad in \quad X.$$

Moreover, the solution U_∞ is globally bounded in X and the energy $t \mapsto E(U_\infty(t))$ is constant.

Proof. By Duhamel's formula we have

$$U_n(t_n) = e^{t_n(\mathbf{A} + \mathbf{B})} U_n(0) + \int_0^{t_n} e^{(t_n - s)(\mathbf{A} + \mathbf{B})} F(U_n(s)) ds.$$

By Theorem 2.9, the semigroup decays exponentially and thus the sequence $e^{t_n(\mathbf{A} + \mathbf{B})} U_n(0)$ goes to zero in X. Second, we need to show that the nonlinear term $\int_0^{t_n} e^{(t_n - s)(\mathbf{A} + \mathbf{B})} F(U_n(s)) ds$ is compact. Using the energy estimate (2.3), we readily see that $U_n(s)$ is uniformly bounded in X for $s \geq 0$. For $\sigma \in [0, 1)$ fixed from now onward, due to Theorem 2.4, $F(U_n(s))$ belongs to a bounded set of X^σ, say, to a ball of radius $R = R(\sigma) > 0$. Since the semigroup decays exponentially, $e^{(t_n - s)(\mathbf{A} + \mathbf{B})} F(U_n(s))$ has an integral in $[0, t_n]$ and thus is uniformly bounded in X^σ as well. Indeed, for each $n \in \mathbb{N}$,

$$\left\| \int_0^{t_n} e^{(t_n - s)(\mathbf{A} + \mathbf{B})} F(U_n(s)) ds \right\|_{X^\sigma} \leq \int_0^{t_n} \|e^{(t_n - s)(\mathbf{A} + \mathbf{B})}\|_{\mathcal{L}(X^\sigma)} \|F(U_n(s))\|_{X^\sigma} ds$$

$$\leq \frac{R}{\lambda} (1 - e^{-\lambda t_n}),$$

from which desired bound follows. By Rellich-Kondrachov X^σ is compactly embedded in X, and thus the sequence $\int_0^{t_n} e^{(t_n - s)(\mathbf{A} + \mathbf{B})} F(U_n(s)) ds$ is compact in X. By diagonal extraction, we get a subsequence $(U_{n_k}(t_{n_k}))_k$ that converges to some $U_\infty(0)$ in X. Let $U_\infty(t) = (u_\infty, \partial_t u_\infty)(t)$ be the maximal solution of the damped plate equation (1.2) with initial data $U_\infty(0)$.

Let $t \in \mathbb{R}$ and let $n \in \mathbb{N}$ large enough such that $t_{n_k} + t \geq 0$. Hence, $U_{n_k}(t_{n_k} + t)$ is well-defined and, moreover, the sequence $(U_{n_k}(t_{n_k} + \cdot))_n$ is uniformly bounded in X. Since our equation is wellposed, the solution is continuous with respect to the initial data. Therefore, from $U(t_{t_{n_k}}) \to U_\infty(0)$ in X as $k \to +\infty$, we obtain that $U(t_{t_{n_k}+t}) \to U_\infty(t)$ in X as $k \to +\infty$, for each t such that $U_\infty(t)$ is well-defined. Moreover, $U_\infty(t)$ is uniformly bounded as well and it can be extended to a global solution $U_\infty(t)$ for $t \in \mathbb{R}$.

Finally, for any $t \in \mathbb{R}$ we have the identity

$$E(U_{n_k}(t_{n_k} + t)) - E(U_{n_k}(t_{n_k})) = - \int_{t_{n_k}}^{t_{n_k}+t} \|\gamma \partial_t u_{n_k}\|_{L^2(\mathcal{M})}^2 dt.$$

Using that the energy functional is continuous, non-increasing, and bounded by above and below, we must have $E(U_{n_k}(t_{n_k} + t)) - E(U_{n_k}(t_{n_k})) \to 0$ as $k \to \infty$, and thus in the limit $E(U_\infty(t)) - E(U_\infty(0)) = 0$, proving that the energy is constant. □

Remark 4.3. Note that for any $T > 0$ we showed that $\int_0^T \|\gamma \partial_t u_\infty\|_{L^2(\mathcal{M})}^2 dt = 0$ and thus $\partial_t u_\infty = 0$ in $(0, T) \times \omega$, which is expected for this kind of trajectories.

4.2.2. *Global attractor.* Here we hinge on the previous results to show the existence of a compact global attractor.

Proof of Theorem 1.4. From Theorem B.8 there are four properties to be verified. First, the asymptotic smoothness of the system was already established in Theorem 4.2.

The positive trajectories of bounded sets are bounded. This property follows as we did in the well-posedness result (2.1). We have that, for any $U \in X$ it holds

$$E(U) \geq \frac{1}{2}\|U\|_X^2 + \text{meas}(\mathcal{M}) \inf_{(x,s) \in \mathcal{M} \times \mathbb{R}} V(x, s).$$

Since the energy is non-increasing, by composition estimates for f, the result follows.

The dynamical system is gradient. From the identity (1.4) is clear that the energy satisfies $E(U(t)) \leq E(U_0)$ for every $t \in \mathbb{R}$. Moreover, the relation $E(U(t)) = E(U_0)$ for every $t \in \mathbb{R}$ means that, for every $T > 0$ it holds

$$\int_0^T \|\gamma(x)\partial_t u\|_{L^2(\mathcal{M})}^2 dt = 0.$$

Therefore, for every $T > 0$ one has

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = 0 & (t, x) \in [0, T] \times \text{Int}(\mathcal{M}), \\ u_{|\partial \mathcal{M}} = \Delta u_{|\partial \mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial \mathcal{M}, \\ \partial_t u = 0 & (t, x) \in [0, T] \times \widetilde{\omega}. \end{cases} \tag{4.7}$$

where $\widetilde{\omega} := \{x \in \mathcal{M} \mid \gamma(x) > \gamma_0/2\}$ is a nonempty open set satisfying $\omega \Subset \widetilde{\omega}$. Since ω satisfies Assumption 1 in time $T > T_0$, we are in position to apply the unique continuation result Theorem 1.5, from which follows that u is an equilibrium point. This combined with the previous item, gives us that the system is gradient.

The set of equilibrium points is bounded. If e is an equilibrium point, by multiplying the equation by e and integrating by parts, we get

$$\frac{1}{2} \int_{\mathcal{M}} \left(|\Delta_g e|^2 + \beta|e|^2\right) dx = -\int_{\mathcal{M}} f(x, e)e \, dx \leq -\text{meas}(\mathcal{M}) \inf_{(x,s) \in \mathcal{M} \times \mathbb{R}} f(x, s)s.$$

Here, we have bounded by below f as in Theorem 2.1. □

5. SEMIGLOBAL CONTROLLABILITY

In this section we aim to prove Theorem 1.3 when the defocusing assumption is released by an asymptotic defocusing assumption (1.6). As showed in Section 4.2, there may exist several equilibrium points and stabilization to zero cannot be expected. To deal with this issue, we follow Joly and Laurent [JL14] who introduced a control technique that allows us to travel through the attractor, exploiting local control around equilibria and the dynamics of the damped plate equation.

5.1. **Local controllability.** We now state a local controllability result near equilibrium points, whose proof is given in Section 5.1.2 below.

Theorem 5.1. *Let f be analytic in the second variable and satisfying the asymptotic defocusing condition* (1.6). *Let $\omega \Subset \widetilde{\omega}$ be two nonempty open sets such that ω satisfies Assumption 1 for some $T_0 > 0$. For $T > T_0$ and for any equilibrium $(e, 0) \in X$ of* (1.1), *there exists a neighborhood $\mathcal{N}(e, 0)$ of $(e, 0)$ such that, for any (u_0, u_1) and (u_0^T, u_1^T) in $\mathcal{N}(e, 0)$, there exists a control $g \in L^\infty([0, 2T], L^2(\mathcal{M}))$ supported in $[0, T] \times \widetilde{\omega}$ such that the unique solution u of*

$$\begin{cases} \partial_t^2 u + \Delta_g^2 u + \beta u + f(x, u) = \mathbb{1}_{\widetilde{\omega}} g & (t, x) \in [0, 2T] \times \mathrm{Int}(\mathcal{M}), \\ u_{|\partial \mathcal{M}} = \Delta u_{|\partial \mathcal{M}} = 0 & (t, x) \in [0, 2T] \times \partial \mathcal{M}, \\ (u, \partial_t u)(0) = (u_0, u_1) & x \in \mathcal{M}, \end{cases} \tag{5.1}$$

satisfies $(u, \partial_t u)(T) = (u_0^T, u_1^T)$.

Let $(e, 0) \in H_D^2 \times L^2(\mathcal{M})$ be an equilibrium of (1.1) and set $z = u - e$. Note that $z \in C^0([0, T], H_D^2) \cap C^1([0, T], L^2(\mathcal{M}))$ and it satisfies

$$\begin{cases} \partial_t^2 z + \Delta_g^2 z + \beta z + \mathfrak{f}(x, z) = g & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}), \\ z_{|\partial \mathcal{M}} = \Delta z_{|\partial \mathcal{M}} = 0 & (t, x) \in [0, T] \times \partial \mathcal{M}, \\ (z, \partial_t z)(0) = (z_0, z_1) & x \in \mathcal{M}, \end{cases} \tag{5.2}$$

with $(z_0, z_1) \in H_D^2 \times L^2(\mathcal{M})$ and $\mathfrak{f}(x, z) := f(x, z + e) - f(x, e)$. Observe that $\mathfrak{f}(x, 0) = 0$ for every $x \in \mathcal{M}$ which reduces the problem of local control towards an equilibrium in time $T > 0$ to local control towards 0 for the system (5.2) in time $T > 0$. Using the reversibility of the uncontrolled plate equation, it is possible to go from an equilibrium towards any desired target in time $T > 0$, thus explaining that the total control time around equilibria is $2T$. Henceforth, we will focus on the (null) control problem (5.2). As usual in control theory, we first study the associated linear control problem through the HUM method, to then establish the control of (5.2) by a perturbation argument introduced by Zuazua [Zua90].

5.1.1. *Linear control.* With the notations of Section 2, with $L^2(\mathcal{M})$ as pivot space, we can introduce the dual space $H_D^{-2} := \left(H_D^2\right)'$ equipped with the norm $\|\phi\|_{H_D^{-2}} = \|(1 + A_0)^{-1}\phi\|_{L^2}$. We can then introduce $X' = L^2 \times H_D^{-2}$, the dual space to X, equipped with the duality product

$$\langle (\phi_0, \phi_1), (\psi_0, \psi_1) \rangle_{X' \times X} = \langle \phi_1, \psi_0 \rangle_{H_D^{-2} \times H_D^2} - \int_{\mathcal{M}} \phi_0 \psi_1 dx,$$

where $\langle \cdot, \cdot \rangle_{H_D^{-2}, H_D^2}$ denotes the duality pairing between H_D^{-2} and H_D^2.

Let us briefly recall the HUM method. Let us consider the linearized control problem

$$\begin{cases} \partial_t^2 v + \Delta_g^2 v + \beta v + \mathfrak{f}_s'(x, 0)v = \gamma^2(x)g, & (t, x) \in [0, T] \times \mathrm{Int}(\mathcal{M}) \\ v_{|\partial \mathcal{M}} = \Delta v_{|\partial \mathcal{M}} = 0, & (t, x) \in [0, T] \times \partial \mathcal{M}, \\ (v, \partial_t v)(T) = (0, 0). & x \in \mathcal{M}, \end{cases} \tag{5.3}$$

along with the operator

$$\begin{array}{rrcl} \mathcal{S}: & L^2([0, T], L^2(\widetilde{\omega})) & \longrightarrow & H_D^2 \times L^2(\mathcal{M}) \\ & g & \longmapsto & (v, \partial_t v)(0). \end{array}$$

For any $(v_0, v_1) \in H_D^2 \times L^2(\mathcal{M})$, a function $g \in L^2([0, T], L^2(\widetilde{\omega}))$ is a null-control in time $T > 0$ for (5.3) if $(v, \partial_t v)(0) = (v_0, v_1)$. By performing integration by parts we get the following duality characterization: such control exists if and only if

$$\iint_{[0, T] \times \mathcal{M}} \gamma^2(x) g \Phi dx dt = \langle (\Phi_0, \Phi_1), (v_0, v_1) \rangle_{X' \times X}, \tag{5.4}$$

for any $(\Phi_0, \Phi_1) \in L^2(\mathcal{M}) \times H_D^{-2}$, where Φ solves the adjoint problem

$$
\begin{cases}
\partial_t^2 \Phi + \Delta_g^2 \Phi + \beta \Phi + \mathfrak{f}_s'(x, 0)\Phi = 0, & (t, x) \in [0, T] \times \text{Int}(\mathcal{M}), \\
\Phi_{|\partial \mathcal{M}} = \Delta \Phi_{|\partial \mathcal{M}} = 0, & (t, x) \in [0, T] \times \partial \mathcal{M}, \\
(\Phi, \partial_t \Phi)(0) = (\Phi_0, \Phi_1), & x \in \mathcal{M}.
\end{cases}
\tag{5.5}
$$

Remark 5.2. Linear wellposedness theory tell us that $(\Phi, \partial_t \Phi) \in C^0([0, T], X')$ is the only solution of (5.5) such that the linear flow map $(\Phi_0, \Phi_1) \in X' \mapsto (\Phi, \partial_t \Phi) \in C^0([0, T], X')$ is bounded. See for instance [LM72, Section 3].

In this direction, we identify the adjoint operator to $\mathcal{S}$,

$$
\begin{array}{cccc}
\mathcal{S}^*: & L^2(\mathcal{M}) \times H_D^{-2} & \longrightarrow & L^2([0, T], L^2(\widetilde{\omega})) \\
& (\Phi_0, \Phi_1) & \longmapsto & \gamma^2(x)\Phi.
\end{array}
$$

The HUM operator is given by $\Lambda := \mathcal{S}\mathcal{S}^*$ as a map from $L^2 \times H_D^{-2}$ into $H_D^2 \times L^2(\mathcal{M})$, defined by

$$
\Lambda(\Phi_0, \Phi_1) = (v, \partial_t v)(0).
$$

If Λ is an isomorphism, then for any initial data $(v, \partial_t v)(0) = (v_0, v_1) \in H_D^2 \times L^2(\mathcal{M})$, the optimal control that steers (v_0, v_1) to $(0, 0)$ is given by $g = \mathcal{S}\Lambda^{-1}(v_0, v_1)$, which also belongs to $L^\infty([0, T], L^2(\mathcal{M}))$. Note that such control is just the localization through $\gamma^2(x)$ to $(0, T) \times \widetilde{\omega}$ of the adjoint solution Φ to (5.5) with corresponding initial data $(\Phi_0, \Phi_1) = \Lambda^{-1}(v_0, v_1)$. With this at hand, the duality identity (5.4) reads as

$$
\iint_{[0,T] \times \mathcal{M}} |\gamma(x)\Phi|^2 dx dt = \langle (\Phi_0, \Phi_1), \Lambda(\Phi_0, \Phi_1) \rangle_{X' \times X},
$$

and thus Λ is an isomorphism if the following observability inequality holds

$$
\|(\Phi_0, \Phi_1)\|_{L^2 \times H_D^{-2}}^2 \leq C \int_0^T \|\gamma \Phi\|_{L^2(\mathcal{M})}^2 dt,
\tag{5.6}
$$

for every $(\Phi_0, \Phi_1) \in L^2 \times H_D^{-2}$. For more details about the HUM operator, we refer for instance to [DL09] for the case of the wave equation.

Having described the HUM method, we point out right away that the exact controllability of the (compactly) perturbed system (5.2) is a consequence of [DO18, Corollary 1.3]. Its verification follows as in Theorem 2.6 above: the observability inequality for the free plate equation in $L^2 \times H_D^{-2}$ is obtained by employing Lebeau's strategy in the corresponding Sobolev space (see [LL24, Proposition 4.3]), and the unique continuation with potential follows similarly as well. In particular, the following result is justified.

Proposition 5.3. *Let ω be a nonempty open set satisfying Assumption 1 for some $T_0 > 0$. For any $T > T_0$, then Λ is an isomorphism.*

5.1.2. *Nonlinear control.* Here we employ a classical strategy that consists in solving the control problem as a perturbation of the corresponding linear one.

Proof of Theorem 5.1. As previously explained, by time reversibility of the uncontrolled plate, in what follows we just focus on the null controllability of (5.2) in time $T > T_0$.

Our control g will be of the form $\gamma^2(x)\Phi$ where Φ is the solution of the free plate equation as in linear control theory with initial datum $(\Phi_0, \Phi_1) \in L^2 \times H_D^{-2}$. We thus consider the solution of the adjoint linear problem

$$
\begin{cases}
\partial_t^2 \Phi + \Delta_g^2 \Phi + \beta \Phi + \mathfrak{f}_s'(x, 0)\Phi = 0, & (t, x) \in [0, T] \times \text{Int}(\mathcal{M}), \\
\Phi_{|\partial \mathcal{M}} = \Delta \Phi_{|\partial \mathcal{M}} = 0, & (t, x) \in [0, T] \times \partial \mathcal{M}, \\
(\Phi, \partial_t \Phi)(0) = (\Phi_0, \Phi_1), & x \in \mathcal{M},
\end{cases}
$$

and the nonlinear control problem

$$\begin{cases} \partial_t^2 z + \Delta_g^2 z + \beta z + \mathfrak{f}(x,z) = \gamma^2(x)\Phi, & (t,x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ z_{|\partial\mathcal{M}} = \Delta z_{|\partial\mathcal{M}} = 0, & (t,x) \in [0,T] \times \partial\mathcal{M}, \\ (z,\partial_t z)(T) = (0,0), & x \in \mathcal{M}. \end{cases}$$

Step 1. Splitting. Let us introduce the operator

$$\begin{array}{rccc} \Lambda_{\text{Nlin}} : & L^2 \times H_D^{-2} & \longrightarrow & H_D^2 \times L^2 \\ & (\Phi_0,\Phi_1) & \longmapsto & \Lambda_{\text{Nlin}}(\Phi_0,\Phi_1) := (z,\partial_t z)(0). \end{array}$$

We aim to perform a perturbative argument. Let us split $z = v + \Psi$, where Ψ is solution of

$$\begin{cases} \partial_t^2 \Psi + \Delta_g^2 \Psi + \beta\Psi + \mathfrak{f}_s'(x,0)\Psi = \gamma^2(x)\Phi, & (t,x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ \Psi_{|\partial\mathcal{M}} = \Delta\Psi_{|\partial\mathcal{M}} = 0, & (t,x) \in [0,T] \times \partial\mathcal{M}, \\ (\Psi,\partial_t\Psi)(T) = (0,0), & x \in \mathcal{M}. \end{cases}$$

This corresponds to the linear control and $(\Psi,\partial_t\Psi)(0) = \Lambda(\Phi_0,\Phi_1)$. The function v is solution of

$$\begin{cases} \partial_t^2 v + \Delta_g^2 v + \beta v + \mathfrak{f}_s'(x,0)v = -R(z) & (t,x) \in [0,T] \times \text{Int}(\mathcal{M}), \\ v_{|\partial\mathcal{M}} = \Delta v_{|\partial\mathcal{M}} = 0 & (t,x) \in [0,T] \times \partial\mathcal{M}, \\ (v,\partial_t v)(T) = (0,0) & x \in \mathcal{M}. \end{cases}$$

with $R(z) = \mathfrak{f}(x,z) - \mathfrak{f}_s'(x,0)z$. Since $\Phi \in C([0,T],L^2(\mathcal{M}))$ we readily get that z, v and Ψ belong to $C([0,T],H_D^2) \cap C^1([0,T],L^2(\mathcal{M}))$. We can write

$$\Lambda_{\text{Nlin}}(\Phi_0,\Phi_1) = \mathcal{K}(\Phi_0,\Phi_1) + \Lambda(\Phi_0,\Phi_1),$$

where $\mathcal{K}(\Phi_0,\Phi_1) = (v,\partial_t v)(0) \in H_D^2 \times L^2(\mathcal{M})$. Note that $\Lambda_{\text{Nlin}}(\Phi_0,\Phi_1) = (z_0,z_1)$ is equivalent to $(\Phi_0,\Phi_1) = -\Lambda^{-1}\mathcal{K}(\Phi_0,\Phi_1) + \Lambda^{-1}(z_0,z_1)$. Let us define the operator $\mathcal{B} : L^2 \times H_D^{-2} \longrightarrow L^2 \times H_D^{-2}$ by

$$\mathcal{B}(\Phi_0,\Phi_1) = -\Lambda^{-1}\mathcal{K}(\Phi_0,\Phi_1) + \Lambda^{-1}(z_0,z_1).$$

Thus the nonlinear problem $\Lambda_{\text{Nlin}}(\Phi_0,\Phi_1) = (z_0,z_1)$ is equivalent to find a fixed point of $\mathcal{B}$.

Step 2. Fixed point estimates. We first verify that the operator $\mathcal{B}$ is compact. This in turns boils down to prove that the operator $\mathcal{K}$ from $L^2 \times H_D^{-2}$ onto its dual is compact. Indeed, exploiting the regularity of the nonlinearity Theorem 2.4 we readily get that $R(z) \in L^1([0,T],H_D^\varepsilon)$ for some $\varepsilon > 0$, which lead us to $v \in C^0([0,T],H_D^{2+\varepsilon}) \cap C^1([0,T],H_D^\varepsilon)$ with a bound on v in such space in terms of $\|(\Phi_0,\Phi_1)\|_{L^2 \times H_D^{-2}}$. First, since $\mathfrak{f}(x,0) = 0$ on $\partial\mathcal{M}$, from the regularity estimate (2.5), $\|R(z)\|_{L^1([0,T],H_D^\varepsilon)}$ is bounded by a function of $\|(z,\partial_t z)\|_{C^0([0,T],X)}$, which in turns is bounded by a function of $\|\Phi\|_{L^1([0,T],L^2(\mathcal{M}))}$ as a consequence of the well-posedness estimates of Theorem 2.1, from which linear energy estimates ensure the veracity of the claim.

Aiming to use Schauder's fixed point theorem, we need to find $R > 0$ such that

$$\|\mathcal{B}(\Phi_0,\Phi_1)\|_{L^2 \times H_D^{-2}} \leq R, \ \forall (\Phi_0,\Phi_1) \in \mathbb{B}_R(L^2 \times H_D^{-2}).$$

Since Λ is an isomorphism, we have

$$\|\mathcal{B}(\Phi_0,\Phi_1)\|_{L^2 \times H_D^{-2}} \leq C\big(\|\mathcal{K}(\Phi_0,\Phi_1)\|_{H_D^2 \times L^2} + \|(z_0,z_1)\|_{H_D^2 \times L^2}\big).$$

To estimate $\mathcal{K}(\Phi_0,\Phi_1)$, we look at the equation satisfied by v, by energy estimates and Theorem A.2,

$$\|(v,\partial_t v)(0)\|_{H_D^2 \times L^2} \leq C\|R(z)\|_{L^1([0,T],L^2(\mathcal{M}))} \leq C\|z\|_{L^\infty([0,T]\times\mathcal{M})}\|(z,\partial_t z)\|_{C^0([0,T],H_D^2 \times L^2)}.$$

From wellposedness estimates for the nonlinear system satisfied by z (see Theorem 2.1 and Theorem 5.2),

$$\|(z,\partial_t z)\|_{C^0([0,T],H_D^2 \times L^2)} \leq C\|\gamma\Phi\|_{L^1([0,T],L^2(\mathcal{M}))} \leq C\|(\Phi_0,\Phi_1)\|_{L^2 \times H_D^{-2}}.$$

From this and Sobolev embedding, it follows

$$\|\mathcal{B}(\Phi_0, \Phi_1)\|_{L^2 \times H_D^{-2}} \leq C\|z\|_{L^\infty([0,T] \times \mathcal{M})}\|(\Phi_0, \Phi_1)\|_{L^2 \times H_D^{-2}} + C\|(z_0, z_1)\|_{H_D^2 \times L^2}$$

$$\leq C\|(\Phi_0, \Phi_1)\|_{L^2 \times H_D^{-2}}^2 + C\|(z_0, z_1)\|_{H_D^2 \times L^2}.$$

Thus, provided that $\|(z_0, z_1)\|_{H_D^2 \times L^2}$ is small enough, there exists $R > 0$ such that $\mathcal{B}$ reproduces the ball of radius $R > 0$ in X'. This completes the proof. $\square$

5.2. Semiglobal controllability.

As already pointed out, we follow the proof of [JL14]. In what follows, given U_0 and U_1 in X, we will write $U_0 \rightsquigarrow U_1$, to say that, there exist $T > 0$ and a control $g \in L^1([0,T], L^2(\mathcal{M}))$ such that the unique solution of (1.5) with initial data U_0 satisfies $(u, \partial_t u)(T) = U_1$. Whenever is needed, the control g and the amount of time spent T will be explicit.

In the following, we summarize some control properties we will use throughout the following proofs.

Property 1. By Theorem 5.1, the nonlinear system (1.5) is locally controllable in a neighborhood $\mathcal{N}(e, 0)$ of an equilibrium point $(e, 0)$. Up to shrinking the neighborhood, we can assume without loss of generality that $\mathcal{N}(e, 0)$ is a ball.

Property 2. If $U(t) = (u, \partial_t u)(t)$ is solution to the damped equation (1.2), by translation in time, then $U(t_0) \rightsquigarrow U(t_0 + T)$ with control $g_+(t, \cdot) = -\gamma \partial_t u(t_0 + t, \cdot)$ acting on $[0, T]$.

Property 3. By reversing the flow, $(u, -\partial_t u)(t_0) \rightsquigarrow (u, -\partial_t u)(t_0 - T)$ with control $g_-(t, \cdot) = \gamma \partial_t u(t_0 - t, \cdot)$ on $[0, T]$.

The following result, called *double U-turn argument* in [JL14], tell us how to travel inside the attractor around equilibrium points. The main idea is that, since the damped plate has a gradient structure, it drives any state toward an equilibrium forward in time and toward another equilibrium backward in time; alternating these dynamics and combining them with local control, we can make two *U-turns* near equilibria and thereby connecting any two points of the trajectory inside the compact attractor.

Proposition 5.4. *If $U(t) = (u, \partial_t u)(t)$ is a solution to the damped plate (1.2) that belongs to the compact global attractor $\mathcal{A}$ for each $t \in \mathbb{R}$, then $U(t_0) \rightsquigarrow U(t_1)$ for any $t_0, t_1 \in \mathbb{R}$.*

Proof. If $t_1 \geq t_0$, we just use Property 2.

If $t_1 < t_0$, by Theorem 1.4 the damped plate (1.2) generates a gradient dynamical system and thus LaSalle's invariance principle Theorem B.7 implies that there exist $t_- < t_1 < t_0 < t_+$ and two equilibrium points e_- and e_+ such that $U(t_\pm)$ belongs to a neighborhood of $(e_\pm, 0)$, denoted by $\mathcal{N}(e_\pm, 0)$. Observe that the forward time exists directly from the gradient structure, whereas the backward time additionally uses the compactness of the attractor. Then, by Property 2, $U(t_0) \rightsquigarrow (u, \partial_t u)(t_+)$ with control $g = -\gamma \partial_t u(t_0 + \cdot, \cdot)$. Since $(u, \partial_t u)(t_+) \in \mathcal{N}(e, 0)$ and such neighborhood is assumed to be a ball centered in $(e_+, 0)$, $(u, -\partial_t u)(t_+)$ also belongs to such neighborhood and by local control Property 1, $(u, \partial_t u)(t_+) \rightsquigarrow (u, -\partial_t u)(t_+)$. By Property 3, $(u, -\partial_t u)(t_+) \rightsquigarrow (u, -\partial_t u)(t_-) \in \mathcal{N}(e_-, 0)$. Once again, as $(u, \partial_t u)(t_-)$ also belongs to $\mathcal{N}(e_-, 0)$, by local control Property 1, $(u, -\partial_t u)(t_-) \rightsquigarrow (u, \partial_t u)(t_-)$. Finally, by Property 2, $(u, \partial_t u)(t_-) \rightsquigarrow U(t_1)$ with control $g = -\gamma^2(x)\partial_t u(t_- + \cdot, \cdot)$. $\square$

5.2.1. Reachable set of an equilibrium. Given $U_0 \in X$, let us introduce the reachable set from U_0, denoted by $\mathcal{R}(U_0)$, as the set of points $U_1 \in X$ such that $U_0 \rightsquigarrow U_1$.

Lemma 5.5. *The reachable set of an equilibrium point $(e, 0)$ is open in X.*

Proof. Let $U_1 \in \mathcal{R}(e,0)$. By definition, $U(0) = (e,0) \rightsquigarrow U(T) = U_1$ with some control $g \in L^1([0,T], L^2(\mathcal{M}))$ in some time $T > 0$. By holding them fixed, the wellposedness result of the controlled plate (1.5) implies that the map $\Psi_{g,T} : X \longrightarrow X$ with $\Psi_{g,T}(U_0) = U(T;g)$ is continuous. Moreover, due to the reversibility in time of the uncontrolled plate equation, $\Psi_{g,T}$ is an homeomorphism in X and thus an open map. In particular, any open neighborhood $\mathcal{W}$ of $(e,0)$ is mapped into an open neighborhood $\mathcal{V} := \Psi_{g,T}(\mathcal{W})$ of U_1. Thus, by definition, for any $U^* \in \mathcal{W}$ and $U_T \in \mathcal{V}$, we have $U^* \rightsquigarrow U_T$ with control g. Further, up to shrinking these neighborhoods if necessary, by local control Property 1 and reversibility of the uncontrolled plate, there exists a local control g_1 such that $(e,0) \rightsquigarrow U^*$. By applying successive controls g_1 and g we have $(e,0) \rightsquigarrow U_T$. This proves that $\mathcal{V} \subset \mathcal{R}(e,0)$ and the conclusion follows. $\qquad\square$

In what follows, for $t \geq 0$ we will denote by S_t the nonlinear semiflow induced by the damped plate equation (1.2). After Theorem 2.1, is clear that $(S_t)_{t \geq 0}$ is a continuous semigroup on X.

Lemma 5.6. *The reachable set restricted to $\mathcal{A}$ of an equilibrium point $(e,0)$, namely $\mathcal{R}(e,0) \cap \mathcal{A}$, is closed in $\mathcal{A}$.*

Proof. Let $(U_1^n)_n$ be a sequence in $\mathcal{A}$ such that $(e,0) \rightsquigarrow U_1^n$ with control g_n on $[0,T_n]$ and $U_1^n \rightarrow U_1$ with $U_1 \in \mathcal{A}$. Let U be solution of the damped plate (1.2) with initial condition $U(0) = U_1$. Since the damped plate has a gradient structure, there exists $T > 0$ large enough and an equilibrium point $(\widetilde{e},0)$ such that $U(T)$ belongs to $\mathcal{N}(\widetilde{e},0)$ neighborhood of $(\widetilde{e},0)$. By continuity of the nonlinear semiflow $(S_t)_{t \geq 0}$ there exists N large enough such that $S_T U_1^N \in \mathcal{N}(\widetilde{e},0)$.

We now apply several successive controls. First, by definition $(e,0) \rightsquigarrow U_1^N$ with g_N. Second, we use Property 2 to get $U_1^N \rightsquigarrow S_T U_1^N$ with control $g = \gamma^2(x)\partial_t u(T_N + \cdot, \cdot)$ and $T > 0$ previously chosen, as we arrive to the neighborhood of an equilibrium. Third, by local control Property 1, we have $S_T U_1^N \rightsquigarrow U(T) \in \mathcal{N}(\widetilde{e},0)$ with $\widetilde{g}$. Finally, given that $U_1 \in \mathcal{A}$ by the U-turn result Theorem 5.4 we have $U(T) \rightsquigarrow U_1$ with $\widetilde{g}_1$. Concatenating these controls, we readily get that $U_1 \in \mathcal{R}(e,0)$ and thus $\mathcal{R}(e,0)$ is closed in $\mathcal{A}$. $\qquad\square$

Proof of Theorem 1.3. Let $(e,0) \in \mathcal{A}$ be an equilibrium point of (1.1), which exists due to the gradient structure of the damped plate and that $\mathcal{A} \neq \emptyset$. After Theorem 5.5 and Theorem 5.6, $\mathcal{R}(e,0) \cap \mathcal{A}$ is both open and closed in $\mathcal{A}$; since the attractor $\mathcal{A}$ (see Theorem B.8) is connected and $\mathcal{R}(e,0) \cap \mathcal{A} \neq \emptyset$, this implies that $\mathcal{R}(e,0) \cap \mathcal{A} = \mathcal{A}$. Hence, by the very definition of reachable set, we can go from the neighborhood of any equilibrium point to the neighborhood of any other equilibrium point.

Step 1. Control strategy. Let U_0 and U_1 in X. Let U and $\widetilde{U}$ be two trajectories of the damped plate (1.2) with $U(0) = U_0$ and $(\widetilde{u}, -\partial_t \widetilde{u})(0) = U_1$. Since the system is gradient, for $T > 0$ large enough, both $U(T)$ and $\widetilde{U}(T)$ belong to neighborhoods of equilibrium points $\mathcal{N}(e_1,0)$ and $\mathcal{N}(e_2,0)$, respectively. In particular, we can assume that these neighborhoods are balls and thus $(\widetilde{u}, -\partial_t \widetilde{u})(T) \in \mathcal{N}(e_2,0)$. The control strategy is the following. First $U_0 \rightsquigarrow U(T)$ arriving to a neighborhood of an equilibrium. Then, since the reachable set of any equilibrium covers $\mathcal{A}$, we travel through the attractor $U(T) \rightsquigarrow (\widetilde{u}, -\partial_t \widetilde{u})(T)$ as both states belong to $\mathcal{A}$. Finally, $(\widetilde{u}, -\partial_t \widetilde{u})(T) \rightsquigarrow U_1$ by using the backward flow.

Step 2. Uniform control time. Regarding the time of controllability, we now show that it only depends on the size of the initial data. From Theorem 5.1, let $\tau > 0$ be the time of local control in a neighborhood of an equilibrium point. Note that is does not depend on the chosen equilibrium point. Also, let us denote by $\mathcal{E}$ the set of equilibria and note that it is compact, since it is closed and contained in the compact attractor $\mathcal{A}$. Let $r(e,0) > 0$ denote the radius of local controllability

around an equilibrium $(e, 0) \in \mathcal{E}$ given by Theorem 5.1. Let us introduce for $U \in \mathcal{A}$, the function

$$\lambda(U) = \inf \left\{ t \geq 0 \mid S_t U \in \bigcup_{(e,0) \in \mathcal{E}} \mathbb{B}_{r(e,0)}(e, 0) \right\},$$

where $\mathbb{B}_{r(e,0)}(e, 0)$ denotes the ball centered at $(e, 0)$ of radius $r(e, 0)$ in X. This quantity is well-defined due to the gradient structure of the damped plate. We claim that there exists $\varepsilon > 0$ such that $T_{\mathcal{A}} := \sup_{U \in \mathcal{V}(\mathcal{A})} \lambda(U) < \infty$ where $\mathcal{V}(\mathcal{A})$ is an ε-neighborhood of $\mathcal{A}$. By contradiction, let us consider the sequences $T_n \to +\infty$ and $(U_0^n)_n \subset X$ with $d(U_0^n, \mathcal{A}) \leq 1/n$, such that $S_t U_0^n$ does not belong to any ball $\mathbb{B}_{r^*}(e, 0)$ for $t \in [0, T_n]$. Since $\mathcal{A}$ is compact, up to extracting a subsequence, we can assume that $U_0^n \to U_0$ with $U_0 \in \mathcal{A}$. Given that S_t is gradient, $S_T U_0 \in \mathbb{B}_{r^*}(e, 0)$ for T large enough. By continuity, $S_T U_0^n \in \mathbb{B}_{r^*}(e, 0)$ for n large enough. This is a contradiction to the definition of the T_n's and that $T_n \to +\infty$.

Let $R > 0$ be the radius of the ball to which the data to be controlled belongs to. By the gradient structure of the damped plate, there exists $T_0 = T_0(R)$ such that $S_{T_0} \mathbb{B}_R(X) \subset \mathcal{V}(\mathcal{A})$. Let us note that the set of equilibrium points $\mathcal{E}$ can be covered by the union of the controllability balls $\mathbb{B}_{r(e,0)}(e, 0)$ where $(e, 0)$ runs over $\mathcal{E}$. By compactness, it can be covered by finitely many of them

$$\mathcal{E} \subset \bigcup_{j=1}^{N} \mathbb{B}_{r(e_j,0)}(e_j, 0),$$

where $(e_j, 0) \in \mathcal{E}$. With this at hand, once we are in $\mathcal{V}(\mathcal{A})$ we wait at most $T_{\mathcal{A}}$ to enter one of the controllability balls $\mathbb{B}_{r(e_j,0)}(e_j, 0)$. The transition between each two balls costs at most $T_{\mathcal{A}} + \tau$ and we can perform this transition at most N times. Then, running the backward flow costs $T_0 + T_{\mathcal{A}}$. We readily see that the total control time is bounded by

$$T_{\max}(R) = 2T_0(R) + 2T_{\mathcal{A}} + N(T_{\mathcal{A}} + \tau),$$

and it only depends on the radius of the given ball. $\qquad \square$

Appendix A. Nonlinear composition estimates

The following lemma establishes the regularity of the composition operator, often called Nemytskii operator, with respect to classical Sobolev spaces.

Lemma A.1. *Let $N \in \mathbb{N}$ and $g : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a $C^{N+1}(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ function, with $g(\cdot, 0) = 0$. If $u \in L^\infty(\mathcal{M}) \cap H^s(\mathcal{M})$, with $s \in (0, N)$, then $g(\cdot, u) \in L^\infty(\mathcal{M}) \cap H^s(\mathcal{M})$ and $\|g(\cdot, u)\|_{H^s} \leq C\|u\|_{H^s}$, where C only depends on g and $\|u\|_{L^\infty}$.*

The previous Lemma is actually written in [AG91] for function in $H^s(\mathbb{R}^d)$ and $f \in C^\infty$, with no extra dependency in the x-variable. However, first, the same result holds for functions in $H^s(\mathcal{M})$ when $\mathcal{M}$ is a compact manifold with boundary using the definition of the norm of $H^s(\mathcal{M})$ by partition of unity and sum of the norm in $H^s(\mathbb{R}^d)$ of the functions in local coordinates and with extension. Second, concerning the requirement $g \in C^{N+1}$, we only notice that the Meyer's multiplier Lemma [AG91, Lemma 2.2] requires estimates of the derivatives of the multiplier up to $\lfloor s \rfloor + 1 \leq N$. Since it is applied to g' in [AG91, Proposition 2.2], it requires $g' \in C^N$. Third, the only change to the proof when the extra x-dependency is added, is that new (mixed) derivatives of g need to be taken into account when analyzing the derivatives of the Meyer's multiplier. However, they are all uniformly bounded by a constant depending on $\|u\|_{L^\infty}$ and on the L^∞-norm of derivatives of g up to order $N + 1$, hence not carrying any dyadic factor depending on the derivative's order at each level. Under these considerations, the proof follows verbatim that of [AG91] and we thus omit it.

Lemma A.2. *Let $g : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathcal{M} \times \mathbb{R})$ function. Then, for any $u \in L^\infty(\mathcal{M}) \cap L^2(\mathcal{M})$ it holds*

$$\|g(\cdot, u) - g(\cdot, 0) - g'_s(\cdot, 0)u\|_{L^2(\mathcal{M})} \leq C\|u\|_{L^\infty(\mathcal{M})}\|u\|_{L^2(\mathcal{M})},$$

where C only depends on g and $\|u\|_{L^\infty}$.

Proof. The estimate follows by writing a second order development

$$g(\cdot, u) - g(\cdot, 0) - g'_s(\cdot, 0)u = \int_0^1 \int_0^\tau g''_s(\cdot, \eta u)u^2 d\eta d\tau,$$

and by using that g''_s is bounded in the compact set $\mathcal{M} \times \{|s| \leq \|u\|_{L^\infty(\mathcal{M})}\}$. □

Appendix B. Dynamical systems

Let X be a Banach space. A continuous dynamical system or continuous semigroup on X, is a one-parameter family of mappings $(S_t)_{t \in \mathbb{R}_*^+}$ from X into X such that

- $S(0) = I$;
- for any $t, s \in \mathbb{R}_*^+$, $S_{t+s} = S_t \circ S_s$;
- for any $t \in \mathbb{R}_*^+$, $S_t \in C^0(X, X)$;
- for any $x_0 \in X$, the application $t \mapsto S_t x_0$ is continuous from $\mathbb{R}_*^+$ to X.

If $\mathbb{R}_*^+$ is replaced by $\mathbb{R}$ above, we say that $(S_t)_{t \in \mathbb{R}}$ is a continuous group on X.

For $x_0 \in X$, we define the positive trajectory $\gamma_+(x_0) = \{S_t x_0, t \geq 0\}$ and the negative trajectory $\gamma_-(x_0) = \{S_t x_0, t \leq 0\}$. To describe the asymptotic behavior we use the concept of an ω-limit set. The set $\omega(x_0) := \bigcap_{t \geq 0} \overline{\{S(s)(x_0), s \geq t\}}$ is called the ω-limit set of trajectories emanating from x_0. This is the set of points $x \in X$ so that there exists a sequence $t_n \to +\infty$ so that $S(t_n)x_0 \to x$. Similarly, the α-limit set is $\alpha(x_0) := \bigcap_{t \leq 0} \overline{\{S(s)(x_0), s \leq t\}}$. This is the set of points $x \in X$ so that there exists a sequence $(t_n)_n$ decreasing to $-\infty$ so that $S(t_n)x_0$ converges to x.

An equilibrium point is a point $e \in X$ so that $S_t e = e$ for any $t \in \mathbb{R}_*^+$. We denote $\mathcal{E}$ the set of equilibrium points.

Definition B.1. A nonempty set $\mathcal{A} \subset X$ is called a global attractor of the semigroup (X, S_t) if:

- $\mathcal{A}$ is a closed, bounded subset of X,
- $\mathcal{A}$ is invariant under the semigroup (X, S_t) (i.e., $S_t \mathcal{A} = \mathcal{A}$ for every $t \geq 0$),
- $\mathcal{A}$ attracts every bounded subset B of X under the group (X, S_t), that is for every $\mathcal{V}$ neighborhood of $\mathcal{A}$, there exists $t > 0$ so that $S_t B \subset \mathcal{V}$.

If it exists, it is given by $\mathcal{A} = \{$bounded complete orbits of $S\}$.

The case where there is a compact global attractor is of particular interest. The existence of such an attractor is an important dynamical property because it roughly says that the dynamics of the PDE for large times may be reduced to dynamics on a compact set, which is often finite-dimensional.

The following properties will be typical of systems that dissipate energy.

Definition B.2. Let (X, S_t) be a dynamical system.

- A closed set $B \subset X$ is said to be *absorbing* for (X, S_t) iff for any bounded set $D \subset X$ there exists $t_0(D)$ such that $S_t D \subset B$ for all $t \geq t_0(D)$.
- (X, S_t) is *point dissipative* iff there exists a bounded set B so that for any $x_0 \in X$, there exists $t_0(x_0) \geq 0$ so that $S_t x_0 \in B$ for all $t \geq t_0(x_0)$,
- (X, S_t) is *asymptotically smooth* iff for any nonempty, closed, bounded set B such that $S_t B \subset B$ for $t > 0$ there exists a compact set $K \subset B$ such that K attracts B; that is

$$\lim_{t \to +\infty} d_X(S_t B, K) = 0,$$

where $d_X(A, B) = \sup_{x \in A} \text{dist}_X(x, B)$.

Remark B.3. Note that the definition of *asymptotic smoothness* is not uniform in the literature, as one may encounter a stronger notion called *asymptotic compactness*. Here, our reference is [Rau02], where there is no distinction in between these two notions.

We have following sequential characterization of asymptotic smoothness, which is often easier to verify than the aforementioned condition.

Proposition B.4. *Assume that the Ladyzhenskaya condition holds: for every bounded sequence $(x_n)_n \subset X$ and every sequence $t_n \to +\infty$, the sequence $(S_{t_n} x_n)_n$ is relatively compact in X. Then (X, S_t) is asymptotically smooth.*

With these definitions at hand, we have the following result ensuring the existence of a compact global attractor.

Theorem B.5. [Rau02, Theorem 2.26] *The following assertions are equivalent.*
- *(X, S_t) admits a global compact attractor $\mathcal{A}$.*
- *(X, S_t) is point dissipative, asymptotically smooth, and for any bounded set $B \subset X$, there exists $t_0 \geq 0$ such that $\overline{\bigcup_{t \geq t_0} S_t B}$ is bounded.*

Moreover, $\mathcal{A}$ is connected.

In practical situations, the previous configuration happens in the presence of some decreasing *energy*.

Definition B.6. A function $\Phi \in C^0(X, \mathbb{R})$ is a called a *strict Lyapunov functional* for (X, S_t) if
- $\Phi(S_t x_0) \leq \Phi(x_0)$ for all $t \geq 0$ and $x_0 \in X$,
- $\Phi(S_t x_0) = \Phi(x_0)$ for all $t \in \mathbb{R}$ implies that x_0 is an equilibrium point.

The dynamical system (X, S_t) is said to be *gradient* if it has a strict Lyapunov functional and each bounded positive orbit is precompact.

Although the definition of a gradient system is not uniform in the literature, all of them enjoy the following property as a consequence.

Theorem B.7 (Invariance principle of LaSalle)**.** *Let (X, S_t) be a gradient system. Then, for any $x_0 \in X$, $\omega(x_0)$ is a set of equilibrium points.*

If x_0 is so that the negative trajectory $\gamma_-(x_0)$ is relatively compact, then $\alpha(x_0)$ is a set of equilibrium points.

Concerning compact attractors, Theorem B.5 simplifies to the following in the context of gradient system.

Theorem B.8. [Rau02, Theorem 4.6] *Let (X, S_t) be a gradient system, so that*
- *(X, S_t) is asymptotically smooth,*
- *for any bounded set $B \subset X$, there exists $t_0 \geq 0$ such that $\bigcup_{t \geq t_0} S_t B$ is bounded,*
- *the set of equilibrium points $\mathcal{E}$ is bounded.*

Then (X, S_t) has a compact global attractor $\mathcal{A}$. Moreover, $\mathcal{A}$ is connected.

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Sorbonne Université, Université Paris Cité, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris, France

Email address: cristobal.loyola@sorbonne-universite.fr