AN EXPLICIT TIME FOR THE UNIFORM NULL CONTROLLABILITY OF A LINEAR KORTEWEG-DE VRIES EQUATION

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Abstract. In this paper, we consider a linear Korteweg-de Vries equation posed in a bounded interval and study the time dependency with respect to the interval length and the transport coefficient, for which the uniform null controllability holds as the dispersion coefficient goes to zero. We consider two cases of boundary controls. First, only one control on the left-end of the interval, and then, two controls acting on the right. The strategy is based on the combination of an exponential dissipation inequality and suitable Carleman estimates for each case.

1. Introduction

1.1. **State of the art of the problem.** Let $T > 0$, $L > 0$ and $Q := (0,T) \times (0,L)$. Consider the following linear Korteweg-de Vries equation posed in a bounded interval

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0, & (t,x) \in Q, \\ y(t,0) = u_1(t), \ y(t,L) = u_2(t), & t \in (0,T), \\ y_x(t,L) = u_3(t), & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $M > 0$ are the dispersion and transport coefficients, respectively, u_1, u_2, u_3 are the controls and y_0 is the initial condition.

The controllability properties of the Korteweg-de Vries equation, which is one of the most important dispersive equations, have been extensively studied over the last two decades. We refer, for instance, to the surveys Cerpa [4], and Rosier and Zhang [18] to consult the main results in this area. Here, we are interested in the uniform null controllability of the linear Korteweg-de Vries equation 1.1 with respect to the dispersion parameter ε. In particular, we search for a time $T_0 > 0$ such that for all $T \geq T_0$ and $y_0 \in L^2(0,L)$, there exists controls u_1, u_2 and u_3 which drive the state y to 0 at $t = T$, while the controls remain uniformly bounded as ε converges to 0^+.

The quantity which measures the cost of the null controllability of 1.1 is defined as the best constant $C_{cost}(\varepsilon)$ such that, for all $y_0 \in L^2(0,L)$ and u_1, u_2, $u_3 \in L^2(0,T)$ driving the solution of 1.1 to 0 at $t = T$, the following inequality holds

$$\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} + \|u_3\|^2_{L^2(0,T)} \leq C_{cost}(\varepsilon) \|y_0\|^2_{L^2(0,L)}.$$

The constant $C_{cost}(\varepsilon)$ is well defined provided that the equation 1.1 is well posed and null controllable, with initial condition $y_0 \in L^2(0,L)$ and controls u_1, u_2, $u_3 \in L^2(0,T)$. It is a classical result that 1.1 is well posed for fixed $\varepsilon > 0$ and $M > 0$ [10]. Likewise, for a fixed $\varepsilon > 0$ the null controllability of 1.1 has been firstly established by Rosier [17] using only one Dirichlet control at $x = 0$ and later improved by Glass and Guerrero [8] requiring less regularity on the initial condition and establishing an upper bound of $C_{cost}(\varepsilon)$.

From a classical property about the null controllability of the transport equation, one would expect that for a time large enough, the cost of the null controllability will decrease to 0 as ε tends to 0^+. Indeed, for the case with one active Dirichlet control at $x = 0$, Glass and Guerrero [10] obtained a uniform upper bound of $C_{cost}(\varepsilon)$ which implies such behavior provided that the time is large enough.

This kind of problems has been first studied in the context of parabolic equations. For the case of a vanishing diffusion coefficient in the heat equation, an explicit time from which the uniform null controllability property holds has been established by Coron and Guerrero [5] by a Carleman estimate approach. Later,

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Glass [9] by a method based on complex analysis improved the previous result. Until now, the best known times, for both explosion and convergence to zero for the cost of null controllability, have been obtained by Lissy [13], [15] where the proof relies on the link between the cost of the null controllability and the cost of fast controls for the heat equation. In the case of higher-order parabolic equations, Carreño and Guzmán [3] studied an equation composed by a transport term, a fourth order term with vanishing viscosity and two boundary controls at the left, establishing an explicit time for which the uniform null controllability holds by using Carleman estimates. By adapting the complex analytic method [9], Lopez-García and Mercado [16] studied a different problem, composed by a transport term and simultaneous vanishing fourth- and third-order terms, using only a single boundary control.

As for the Korteweg-de Vries equation 1.1, the uniform null controllability was proved by Glass and Guerrero [8] using three active boundary controls while the same authors in [10] improved their previous null controllability result by using only one active Dirichlet control at the left of the interval. Then, Carreño and Guerrero [1], [2] obtained some results about the behavior of the cost of the null controllability in the case of Colin-Ghidaglia boundary conditions. In [1], the authors proved that the cost with one control at the left-end point of the interval grows exponentially as ε goes to zero for any $T > 0$, and in [2] a uniform null controllability result is proved in the zero-dispersion limit by using two controls with a restriction on the initial condition.

1.2. **Main results.** The first main result recovers the conclusion of [10, Theorem 1.1] for the case of null right boundary conditions, that is, $u_2 = u_3 = 0$. Nevertheless, here we establish an explicit lower bound for the time for which the uniform null controllability holds.

Theorem 1. *Let $T \geq 23.25L/M$ with $M > 0$ and consider $u_2 = u_3 = 0$. There exists two positive constants C and c independent of T, L, $\varepsilon > 0$ and M such that for any $\varepsilon > 0$ and $y_0 \in L^2(0,L)$ there exists a control $u_1 \in L^2(0,T)$ driving the state y of 1.1 to 0 in time T which can be estimated as follows*

$$\|u_1\|^2_{L^2(0,T)} \leq C\frac{L^5}{\varepsilon^2 T}\exp\left\{-c\frac{L^{3/2}}{\varepsilon^{1/2}T^{1/2}}\right\}\|y_0\|^2_{L^2(0,L)}. \tag{1.2}$$

In this case, the *cost of the null controllability* is defined by

$$C^1_{cost}(\varepsilon) := \sup_{\substack{y_0\in L^2(0,L)\\ y_0\neq 0}} \min_{\substack{u_1\in L^2(0,T)\\ y(T)=0}} \frac{\|u_1\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}}.$$

We obtain the following corollary about the behavior of the cost of null controllability.

Corollary 2. *If $T \geq 23.25L/M$ with $M > 0$, then $C^1_{cost}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.*

If the Dirichlet control at the right is kept null, that is, $u_1 \equiv 0$, a similar null controllability result is obtained.

Theorem 3. *Let $T \geq 24.31L/M$ with $M > 0$ and $u_1 = 0$. There exists two positive constants C and c independent of T, L, $\varepsilon > 0$ and M such that for any $\varepsilon > 0$ and $y_0 \in L^2(0,L)$ there exists controls u_2, $u_3 \in L^2(0,T)$ driving the state y of 1.1 to 0 in time T which can be estimated as follows*

$$\|u_2\|^2_{L^2(0,T)} + \|u_3\|^2_{L^2(0,T)} \leq \frac{C}{\varepsilon^2}\left(\frac{L^3+L^5}{T}\right)\exp\left\{-c\frac{L^{3/2}}{\varepsilon^{1/2}T^{1/2}}\right\}\|y_0\|^2_{L^2(0,L)}. \tag{1.3}$$

Now the *cost of null controllability* is defined by

$$C^2_{cost}(\varepsilon) := \sup_{\substack{y_0\in L^2(0,L)\\ y_0\neq 0}} \min_{\substack{(u_2,u_3)\in L^2(0,T)^2\\ y(T)=0}} \frac{\|u_2\|^2_{L^2(0,T)} + \|u_3\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}}.$$

As expected, we get the following corollary.

Corollary 4. *If $T \geq 24.31L/M$ with $M > 0$, then $C^2_{cost}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.*

From the Hilbert Uniqueness Method (see, for instance, [12], [6]), it is well known that inequalities 1.2 and 1.3 are equivalent to an observability inequality for the solutions of the adjoint system of 1.1. To prove the observability, we combine an appropriate Carleman estimate with an exponential dissipation inequality for the solutions of the adjoint system of 1.1 under the assumption that $T \geq \tau L/M$. The constant appearing in the observability inequality will be of the form $C^* \exp\{K(\varepsilon, L, M, T)\}$. Choosing the weight function of the Carleman estimate in a suitable way and integrating in an appropriate time-cut interval the Carleman estimate, by an optimization problem we obtain an explicit τ^* for which $\exp\{K(\varepsilon, L, M)\} \to 0$ as $\varepsilon \to 0^+$.

1.3. **Organization of the paper.** The rest of the paper is organized as follows. In Section 2, we recall some results concerning the well-posedness and null controllability of equation 1.1. Section 3 is devoted to Carleman estimates. In Section 4, we prove Theorems 1 and 3. Finally, in Section 5 we present an application of the approach of this paper which improves a result from [3].

2. Cauchy problem and controllability

In this section, we recall and formalize the well-posedness and the notion of null controllability for equation (1.1).

2.1. **Well-posedness.** Now we introduce the notion of *solutions by transposition* in order to establish the well-posedness result for the equation (1.1).

Definition 5. Let $T > 0$, $y_0 \in H^{-1}(0, L)$ and $(u_1, u_2, u_3) \in L^2(0, L) \times L^2(0, L) \times H^{-1/3}(0, T)$. We call y a *solution by transposition* of (1.1), to a function $y \in L^2((0, T) \times (0, L))$ satisfying

$$\iint_Q yf\,dxdt = \langle y_0, z_{|t=0}\rangle_{H^{-1}(0,L)\times H^1(0,L)} + \varepsilon \int_0^T u_1 z_{xx|_{x=0}} dt - \varepsilon \int_0^T u_2 z_{xx|_{x=L}} dt + \varepsilon \langle u_3, z_{x|x=L}\rangle_{H^{-1/3}(0,T)\times H^{1/3}(0,T)}, \quad (2.1)$$

for all $f \in L^2((0, T) \times (0, L))$, where $z \in C([0, T]; H_0^1(0, L)) \cap L^2(0, T; H^2(0, L))$ is the solution of

$$\begin{cases} -z_t - \varepsilon z_{xxx} + M z_x = f, & (t, x) \in Q, \\ z(t, 0) = z(t, L) = z_x(t, 0) = 0, & t \in (0, T), \\ z(T, x) = 0, & x \in (0, L). \end{cases} \quad (2.2)$$

The following well-posedness result is established in [8].

Proposition 6. *Let $y_0 \in H^{-1}(0, L)$ and $(u_1, u_2, u_3) \in L^2(0, L) \times L^2(0, L) \times H^{-1/3}(0, T)$. Then there exists a unique solution y of* (1.1) *satisfying*

$$y \in L^2((0, T) \times (0, L)) \cap C([0, T]; H^{-1}(0, L)),$$

and

$$\|y\|_{L^2((0,T)\times(0,L))} + \|y\|_{L^\infty(0,T;H^{-1}(0,L))} \leq \frac{C}{\varepsilon}\big(\|y_0\|_{H^{-1}(0,L)} + \|u_1\|_{L^2(0,T)} + \|u_2\|_{L^2(0,T)} + \|u_3\|_{H^{-1/3}(0,T)}\big),$$

for some $C > 0$ independent of y_0, u_1, u_2, u_3 and ε.

2.2. **Controllability.** The null controllability of system (1.1) can be characterized by the adjoint system, which is given by

$$\begin{cases} -\varphi_t - \varepsilon \varphi_{xxx} + M\varphi_x = 0, & (t, x) \in Q, \\ \varphi(t, 0) = \varphi(t, L) = \varphi_x(t, 0) = 0, & t \in (0, T), \\ \varphi(T, x) = \varphi_T(x), & x \in (0, L), \end{cases} \quad (2.3)$$

where $\varphi_T \in L^2(0, L)$. Then, given $y_0 \in L^2(0, L)$, u_1, u_2, $u_3 \in L^2(0, T)$ are controls such that $y(T, x) = 0$ if and only if

$$\int_0^L y_0 \varphi_{|t=0} dx = -\varepsilon \int_0^T u_1 \varphi_{xx|_{x=0}} dt + \varepsilon \int_0^T u_2 \varphi_{xx|_{x=L}} dt - \varepsilon \int_0^T u_3 \varphi_{x|_{x=L}} dt,$$

for each $\varphi_T \in L^2(0, L)$, where φ is the corresponding solution of (2.3) associated to φ_T.

From the classical duality between controllability and observability, we have the following characterizations for the cases treated in this paper.

Proposition 7. *System* (1.1) *is null controllable with* $u_2 = u_3 = 0$ *if and only if there exists* $C > 0$ *such that*

$$\int_0^L |\varphi_{|t=0}|^2 dx \leq C \int_0^T |\varphi_{xx|_{x=0}}|^2 dt, \tag{2.4}$$

for each $\varphi_T \in L^2(0, L)$*, where* φ *is the solution of the adjoint system* (2.3) *associated to* $\varphi_T \in L^2(0, L)$*.*

As a direct consequence of the previous result, the L^2-norm of the control u_1 can be estimated by

$$\|u_1\|^2_{L^2(0,T)} \leq \frac{C}{\varepsilon^2} \|y_0\|^2_{L^2(0,L)},$$

where C is the same constant obtained in the observability inequality (2.4).

Proposition 8. *System* (1.1) *is null controllable with* $u_1 = 0$ *if and only if there exists* $C > 0$ *such that*

$$\int_0^L |\varphi_{|t=0}|^2 dx \leq C \left(\int_0^T |\varphi_{xx|_{x=L}}|^2 dt + \int_0^T |\varphi_{x|_{x=L}}|^2 dt \right), \tag{2.5}$$

for each $\varphi_T \in L^2(0, L)$*, where* φ *is the solution of the adjoint system* (2.3) *associated to* $\varphi_T \in L^2(0, L)$*.*

As before, the L^2-norm of the controls u_2 and u_3 can be estimated by

$$\|u_2\|^2_{L^2(0,T)} + \|u_3\|^2_{L^2(0,T)} \leq \frac{C}{\varepsilon^2} \|y_0\|^2_{L^2(0,L)},$$

where C is the same constant obtained in the observability inequality (2.5).

Therefore, to prove Theorems 1 and 3, it is sufficient to establish estimates like (2.4) and (2.5), respectively, with a suitable constant C.

3. Carleman estimates for the adjoint system

This section is devoted to two Carleman estimates for the adjoint equation (2.3). These inequalities are essential to prove Theorems 1 and 3.

3.1. Carleman estimate with observation at $x = 0$.

Let us introduce the weight function

$$\alpha(t, x) = \frac{\beta(x)}{t^{1/2}(T - t)^{1/2}}, \ (t, x) \in Q, \tag{3.1}$$

where β is a strictly positive, strictly increasing and concave polynomial of degree 2. To ease the notation we define

$$\rho(t) := t^{-1/2}(T - t)^{-1/2}, \ t \in (0, T).$$

The Carleman estimate is the following one.

Proposition 9. *There exists positive constants* C*,* C_1*,* C_2 *and* C_3 *independent of* ε*,* L*,* s *and* $M \in \mathbb{R}$ *such that for any* $\varphi_T \in L^2(0, L)$ *we have*

$$C \left(\iint_Q L^4 s^5 \rho^5 e^{-2s\alpha} |\varphi|^2 dxdt + \iint_Q L^2 s^3 \rho^3 e^{-2s\alpha} |\varphi_x|^2 dxdt \right) \leq \int_0^T Ls\rho e^{-2s\alpha_{|x=0}} |\varphi_{xx|_{x=0}}|^2 dt, \tag{3.2}$$

for any $s \geq C_1 T^{1/2} L^{-1/2} \varepsilon^{-1/2} + C_2 T L^{-1} \varepsilon^{1/2} |M|^{1/2} + C_3 T L^{-2}$*, where* φ *is the solution of* (2.3) *associated to* φ_T*.*

Remark 10. Although it appears that C_1 and C_2 are fixed here, these constants will play a role as parameters in the proof of Theorem 1.

The inequality (3.2) follows directly from the one developed by Glass and Guerrero [10, Proposition 3.1] by considering the case of null diffusion coefficient and with $\rho(t) = t^{-\mu}(T - t)^{-\mu}$ for $\mu \in [1/2, 1]$. Nevertheless, here we choose explicitly the power $\mu = 1/2$, which is the optimal one for the Korteweg-de Vries equation and crucial for the proof of the Theorem 1.

Proof of Proposition 9. Now we shall prove the Carleman estimate (3.2) using a standard procedure due to Fursikov and Imanuvilov [7].

Weight function. Consider α as in (3.1) and let $\beta(x) = -ax^2 + bLx + cL^2$ be an strictly positive, strictly increasing and concave polynomial of degree 2. Then a, b and c are positive real numbers and α must satisfy the following inequalities in Q

$$L^2 c\rho \leq \alpha \leq L^2(-a+b+c)\rho,\ L(b-2a)\rho \leq \alpha_x \leq Lb\rho,\ \alpha_{xx} = -2a\rho.$$

In addition we impose that $b > 2a$. Concerning to the time derivatives, by using $2 \leq T\rho$ we get the following estimates

$$|\alpha_t| \leq \frac{1}{2}(b+c-a)TL^2\rho^3,\ |\alpha_{xt}| \leq \frac{b}{2}TL\rho^3, |\alpha_{xxt}| \leq aT\rho^3,\ |\alpha_{tt}| \leq \frac{3}{4}(b+c-a)T^2L^2\rho^5.$$

Conjugate operator. Let us define $\mathcal{L}\varphi := \varphi_t + \varepsilon\varphi_{xxx} - M\varphi_x$. For $s > 0$, set $\psi = e^{-s\alpha}\varphi$ and introduce the conjugate operator $\mathcal{P}\psi = e^{-s\alpha}\mathcal{L}(e^{s\alpha}\psi)$. Consider the decomposition of $\mathcal{P}\psi = \mathcal{L}_1\psi + \mathcal{L}_2\psi + \mathcal{R}\psi$ given by

$$\begin{aligned}
\mathcal{L}_1\psi &= \varepsilon\psi_{xxx} + \psi_t + 3\varepsilon s^2(\alpha_x)^2\psi_x - M\psi_x,\\
\mathcal{L}_2\psi &= \varepsilon s^3(\alpha_x)^3\psi + 3\varepsilon s\alpha_x\psi_{xx} + s\alpha_t\psi + 3\varepsilon s\alpha_{xx}\psi_x - Ms\alpha_x\psi,\\
\mathcal{R}\psi &= \varepsilon s\alpha_{xxx}\psi + 3\varepsilon s^2\alpha_x\alpha_{xx}\psi.
\end{aligned}$$

Taking the L^2-norm in Q to $\mathcal{L}_1\psi + \mathcal{L}_2\psi = \mathcal{P}\psi - \mathcal{R}\psi$ we obtain

$$\|\mathcal{L}_1\psi\|^2_{L^2(Q)} + \|\mathcal{L}_2\psi\|^2_{L^2(Q)} + 2\iint_Q \mathcal{L}_1\psi\mathcal{L}_2\psi dxdt = \|\mathcal{P}\psi - \mathcal{R}\psi\|^2_{L^2(Q)},$$

from which it follows

$$\iint_Q \mathcal{L}_1\psi\mathcal{L}_2\psi dxdt \leq \|\mathcal{P}\psi\|^2_{L^2(Q)} + \|\mathcal{R}\psi\|^2_{L^2(Q)}. \tag{3.3}$$

Computation of the double product term. Let us denote by I_{ij} for $1 \leq i \leq 4, 1 \leq j \leq 5$ the L^2-product in Q between the ith term of $\mathcal{L}_1\psi$ with the jth term of $\mathcal{L}_2\psi$. Integration by parts are performed and each resulting expression for I_{ij} is listed below.

First, we will compute $\langle(\mathcal{L}_1\psi)_1, \mathcal{L}_2\psi\rangle_{L^2(Q)}$.

- Using the null boundary conditions and the fact that $\alpha_{xxx} = 0$

$$I_{11} = \frac{9}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2 dxdt - \varepsilon^2\frac{s^3}{2}\int_0^T \alpha^3_{x|_{x=L}}|\psi_{x|_{x=L}}|^2 dxdt - 3\varepsilon^2 s^3 \iint_Q \alpha^3_{xx}|\psi|^2 dxdt.$$

- Here we only integrate by parts

$$I_{12} = -\frac{3}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt + \frac{3}{2}\varepsilon^2 s\int_0^T \alpha_{x|_{x=L}}|\psi_{xx|_{x=L}}|^2 dt - \frac{3}{2}\varepsilon^2 s\int_0^T \alpha_{x|_{x=0}}|\psi_{xx|_{x=0}}|^2 dt.$$

- Again, using the null boundary conditions and that $\alpha_{xxx} = 0$

$$I_{13} = \frac{3}{2}\varepsilon s \iint_Q \alpha_{xt}|\psi_x|^2 dxdt - \varepsilon\frac{s}{2}\int_0^T \alpha_{t|_{x=L}}|\psi_{x|_{x=L}}|^2 dt.$$

- Integrating by parts, using that $\alpha_{xxx} = 0$ and $\varphi_{|_{x=0}} = \varphi_{x|_{x=0}} = 0$ we get

$$I_{14} = -3\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt + 3\varepsilon^2 s\int_0^T \alpha_{xx|_{x=L}}\psi_{xx|_{x=L}}\psi_{x|_{x=L}} dt.$$

- Using that $\alpha_{xxx} = 0$, we compute the last term

$$I_{15} = -\frac{3}{2}\varepsilon sM \iint_Q \alpha_{xx}|\psi_x|^2 dxdt + \varepsilon s\frac{M}{2}\int_0^T \alpha_{x|_{x=L}}|\psi_{x|_{x=L}}|^2 dt.$$

Now we will concern about the term $\langle(\mathcal{L}_1\psi)_2, \mathcal{L}_2\psi\rangle_{L^2(Q)}$.

- Integrating by parts with respect to t we get

$$I_{21} = -\varepsilon\frac{3s^3}{2}\iint_Q \alpha_x^2\alpha_{xt}|\psi|^2 dxdt.$$

- Here using that $\psi_{x|_{t=0}} = \psi_{x|_{t=T}} = 0$, we obtain

$$I_{22} = \varepsilon\frac{3s}{2}\iint_Q \alpha_{xt}|\psi_x|^2 dxdt - 3\varepsilon s\iint_Q \alpha_{xx}\psi_x\psi_t dxdt.$$

- Again using that $\psi_{|_{t=0}} = \psi_{|_{t=T}} = 0$ we have

$$I_{23} = -s\iint_Q \alpha_{tt}|\psi|^2 dxdt.$$

- The following term will cancel out with the last term of I_{22}

$$I_{24} = 3\varepsilon s\iint_Q \alpha_{xx}\psi_t\psi_x dxdt.$$

- Again, we use $\psi_{|_{t=0}} = \psi_{|_{t=T}} = 0$ to obtain

$$I_{25} = \frac{Ms}{2}\iint_Q \alpha_{xt}|\psi|^2 dxdt$$

Now we compute the third term $\langle(\mathcal{L}_1\psi)_3, \mathcal{L}_2\psi\rangle_{L^2(Q)}$.

- Using the null boundary conditions

$$I_{31} = -\frac{15}{2}\varepsilon^2 s^5\iint_Q (\alpha_x)^4\alpha_{xx}|\psi|^2 dxdt.$$

- Using that $\psi_{x|_{x=0}} = 0$

$$I_{32} = -\frac{27}{2}\varepsilon^2 s^3\iint_Q (\alpha_x)^2\alpha_{xx}|\psi_x|^2 dxdt + \frac{9}{2}\varepsilon^2 s^3\int_0^T \alpha^3_{x|_{x=L}}|\psi_{x|_{x=L}}|^2 dt.$$

- From the identity $(\alpha_t\alpha_x^2)_x = \alpha_{tx}\alpha_x^2 + 2\alpha_t\alpha_x\alpha_{xx}$ and integration by parts we obtain

$$I_{33} = -\frac{3}{2}\varepsilon s^3\iint_Q \alpha_{xt}\alpha_x^2|\psi|^2 dxdt - \frac{3}{2}\varepsilon s^3\iint_Q \alpha_t\alpha_x\alpha_{xx}|\psi|^2 dxdt.$$

- The fourth term is given by

$$I_{34} = 9\varepsilon^2 s^3\iint_Q (\alpha_x)^2\alpha_{xx}|\psi_x|^2 dxdt.$$

- Finally

$$I_{35} = \frac{9}{2}M\varepsilon s^3\iint_Q (\alpha_x)^2\alpha_{xx}|\psi|^2 dxdt.$$

Now, we compute inner product of the fourth term $\langle(\mathcal{L}_1\psi)_4, \mathcal{L}_2\psi\rangle_{L^2(Q)}$.

- We integrate by parts once to obtain

$$I_{41} = -\frac{3}{2}\varepsilon s^3 M\iint_Q (\alpha_x)^2\alpha_{xx}|\psi|^2 dxdt.$$

- Integrating by parts and using $\varphi_{x|_{x=0}} = 0$

$$I_{42} = \frac{3}{2}\varepsilon sM\iint_Q \alpha_{xx}|\psi_x|^2 dxdt - \frac{3}{2}\varepsilon sM\int_0^T \alpha_{x|_{x=L}}|\psi_{x|_{x=L}}|^2 dt$$

- Using the null boundary conditions

$$I_{43} = \frac{s}{2}M\iint_Q \alpha_{tx}|\psi|^2 dxdt.$$

- The fourth term is given by

$$I_{44} = -3\varepsilon sM\iint_Q \alpha_{xx}|\psi_x|^2 dxdt$$

- Finally

$$I_{45} = -\frac{s}{2}M^2 \iint_Q \alpha_{xx}|\psi_x|^2 dxdt.$$

Gathering the terms. Putting all together the distributed terms and the boundary terms we have

$$\begin{aligned}\mathcal{D}(\psi) := &-\frac{15}{2}\varepsilon^2 s^5 \iint_Q (\alpha_x)^4 \alpha_{xx}|\psi|^2 dxdt - \frac{9}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt \\ &- 3\varepsilon^2 s^3 \iint_Q \alpha_{xx}^3 |\psi|^2 dxdt - 3\varepsilon s^3 \iint_Q \alpha_{tx}\alpha_x^2|\psi|^2 dxdt - \frac{3}{2}\varepsilon s^3 \iint_Q \alpha_t\alpha_x\alpha_{xx}|\psi|^2 dxdt \\ &+ 3M\varepsilon s^3 \iint_Q (\alpha_x)^2\alpha_{xx}|\psi|^2 dxdt - s\iint_Q \alpha_{tt}|\psi|^2 dxdt - Ms\iint_Q \alpha_{tx}|\psi|^2 dxdt \\ &+ 3\varepsilon s \iint_Q \alpha_{xt}|\psi_x|^2 dxdt - \frac{3M}{2}\varepsilon s \iint_Q \alpha_{xx}|\psi_x|^2 dxd6 - \frac{M^2}{2} s \iint_Q \alpha_{xx}|\psi_x|^2 dxdt,\end{aligned}$$

$$\begin{aligned}\mathcal{B}_L(\psi) := {} & 4\varepsilon^2 s^3 \int_0^T \alpha^3_{x|_{x=L}} |\psi_{x|_{x=L}}|^2 dxdt + \frac{3}{2}\varepsilon^2 s \int_0^T \alpha_{x|_{x=L}} |\psi_{xx|_{x=L}}|^2 dt - \varepsilon\frac{s}{2}\int_0^T \alpha_{t|_{x=L}}|\psi_{x|_{x=L}}|^2 dt \\ &+ 3\varepsilon^2 s \int_0^T \alpha_{xx|_{x=L}}\psi_{x|_{x=L}}\psi_{xx|_{x=L}} dt + 2M\varepsilon s \int_0^T \alpha_{x|_{x=L}}|\psi_{x|_{x=L}}|^2 dt\end{aligned}$$

and

$$\mathcal{B}_0(\psi) := -\frac{3}{2}\varepsilon^2 s \int_0^T \alpha_{x|_{x=0}}|\psi_{xx|_{x=0}}|^2 dt.$$

Distributed terms. We shall use systematically the estimates developed in the previous subsection about the weight function α. Concerning the dominating terms of $\mathcal{D}(\psi)$ we have that

$$\begin{aligned}&-\frac{15}{2}\varepsilon^2 s^5 \iint_Q (\alpha_x)^4\alpha_{xx}|\psi|^2 dxdt - \frac{9}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt \\ &\qquad \geq 15a(b-2a)^4 \iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt + 9a\iint_Q \varepsilon^2 s\rho|\psi_{xx}|^2 dxdt. \quad (3.4)\end{aligned}$$

In order to deal with the first order terms above, we integrate by parts and use the null boundary conditions to obtain

$$(b-2a)^2 \iint_Q L^2 s^3\rho^3 \psi_x\psi_x dxdt = -(b-2a)^2 \iint_Q L^2 s^3 \rho^3 \psi_{xx}\psi dxdt$$

By using Young's inequality we obtain that

$$6a(b-2a)^2 \iint_Q L^2 s^3\rho^3|\psi_x|^2 dxdt \leq a(b-2a)^4 \iint_Q L^4 s^5\rho^5|\psi|^2 dxdt + 9a\iint_Q s\rho|\psi_{xx}|^2 dxdt.$$

Then from (3.4) and the above inequality it follows that

$$\begin{aligned}&-\frac{15}{2}\varepsilon^2 s^5 \iint_Q (\alpha_x)^4\alpha_{xx}|\psi|^2 dxdt - \frac{9}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 dxdt \\ &\qquad \geq 14a(b-2a)^4 \iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt + 6a(b-2a)^2 \iint_Q \varepsilon^2 L^2 s^3\rho^3|\psi_x|^2 dxdt.\end{aligned}$$

Since $\rho^{-1} \leq T/2$, for the remaining terms we get

$$-3\varepsilon^2 s^3 \iint_Q \alpha_{xx}^3|\psi|^2 dxdt \geq 0,$$

$$\left|3\varepsilon s^3 \iint_Q \alpha_{tx}\alpha_x^2|\psi|^2 dxdt\right| \leq \frac{3}{2}b^3\frac{T}{L\varepsilon s^2}\iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt,$$

$$\left|\frac{3}{2}\varepsilon s^3 \iint_Q \alpha_t\alpha_x\alpha_{xx}|\psi|^2 dxdt\right| \leq \frac{3}{2}(b+c-a)ab\frac{T}{L\varepsilon s^2}\iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt,$$

$$\left|3M\varepsilon s^3 \iint_Q (\alpha_x)^2\alpha_{xx}|\psi|^2 dxdt\right| \leq \frac{3}{2}ab^2\frac{T^2|M|}{L^2\varepsilon s^2}\iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt,$$

$$\left|s\iint_Q \alpha_{tt}|\psi|^2 dxdt\right| \leq \frac{3}{4}(b+c-a)\frac{T^2}{L^2\varepsilon^2 s^4}\iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt,$$

$$\left|Ms\iint_Q \alpha_{tx}|\psi|^2 dxdt\right| \leq \frac{b}{8}\frac{T^3|M|}{L^3\varepsilon^2 s^4}\iint_Q \varepsilon^2 L^4 s^5\rho^5|\psi|^2 dxdt,$$

$$\left|3\varepsilon s\iint_Q \alpha_{xt}|\psi_x|^2 dxdt\right| \leq \frac{3}{2}b\frac{T}{L\varepsilon s^2}\iint_Q \varepsilon^2 L^2 s^3\rho^3|\psi_x|^2 dxdt,$$

$$\left|\frac{3M}{2}\varepsilon s\iint_Q \alpha_{xx}|\psi_x|^2 dxdt\right| \leq \frac{3}{4}a\frac{T^2|M|}{L^2\varepsilon s^2}\iint_Q \varepsilon^2 L^2 s^3\rho^3|\psi_x|^2 dxdt,$$

$$-\frac{M^2}{2}s\iint_Q \alpha_{xx}|\psi_x|^2 dxdt \geq 0.$$

Gathering the above estimates we obtain

$$\mathcal{D}(\psi) \geq \mathcal{D}_0(s)\varepsilon^2\iint_Q L^4 s^5\rho^5|\psi|^2 dxdt + \mathcal{D}_1(s)\varepsilon^2\iint_Q L^2 s^3\rho^3|\psi_x|^2 dxdt,$$

where

$$\mathcal{D}_0(s) = 14a(b-2a)^4 - \frac{3}{2}b^2\frac{T}{L\varepsilon s^2} - \frac{3}{2}(b+c-a)ab\frac{T}{L\varepsilon s^2} - \frac{3}{2}ab^2\frac{|M|T^2}{L^2\varepsilon s^2} - \frac{3}{4}(b+c-a)\frac{T^2}{L^2\varepsilon^2 s^4} - \frac{b}{8}\frac{T^3|M|}{L^3\varepsilon^2 s^4},$$
$$\mathcal{D}_1(s) = 6a(b-2a)^2 - \frac{3}{2}b\frac{T}{L\varepsilon s^2} - \frac{3}{4}a\frac{T^2|M|}{L^2\varepsilon s^2}.$$

In order to handle $\mathcal{D}_0$ and $\mathcal{D}_1$, from now on we consider s fulfilling

$$s \geq C_1 T^{1/2}L^{-1/2}\varepsilon^{-1/2} + C_2 TL^{-1}\varepsilon^{1/2}|M|^{1/2} + C_3 TL^{-2}, \tag{3.5}$$

where C_1, $C_2 > 0$ and $C_3 > 0$ will be choosen later. Note that this choice of s implies

$$\frac{T}{L\varepsilon s^2} \leq \frac{1}{C_1^2},\ \frac{T^2|M|}{L^2\varepsilon s^2} \leq \frac{1}{C_2^2},\ \frac{T}{L^2 s} \leq \frac{1}{C_3}$$

and hence, we get the following estimates

$$\mathcal{D}_0(s) \geq 14a(b-2a)^4 - \frac{3}{2}b^2\frac{1}{C_1^2} - \frac{3}{2}(b+c-a)ab\frac{1}{C_1^2} - \frac{3}{2}ab^2\frac{1}{C_2^2} - \frac{3}{4}(b+c-a)\frac{1}{C_1^4} - \frac{b}{8}\frac{1}{C_1^2 C_2^2},$$
$$\mathcal{D}_1(s) \geq 6a(b-2a)^2 - \frac{3}{2}b\frac{1}{C_1^2} - \frac{3}{4}a\frac{1}{C_2^2}.$$

Boundary terms. Observe that the term $\mathcal{B}_0$ is precisely the observation at $x=0$, therefore we have to estimate the terms at $x=L$. Concerning to the dominating first order term of $\mathcal{B}_L$ we obtain

$$4\varepsilon^2 s^3 \int_0^T \alpha^3_{x|_{x=L}} |\psi_{x|_{x=L}}|^2 dt \geq 4(b-2a)^3 \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt$$

Now estimating the remaining terms of $\mathcal{B}_L$ we get

$$\left| \varepsilon \frac{s}{2} \int_0^T \alpha_{t|_{x=L}} |\psi_{x|_{x=L}}|^2 dt \right| \leq \frac{1}{4}(b+c-a) \frac{T}{\varepsilon L s^2} \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt.$$

Here we use that $2|uv| \leq u^2+v^2$ to obtain

$$\begin{aligned} \left| 3\varepsilon^2 s \int_0^T \alpha_{xx|_{x=L}} \psi_{x|_{x=L}} \psi_{xx|_{x=L}} dt \right| &\leq 3\varepsilon^2 s \int_0^T \frac{|\alpha_{xx|_{x=L}}|}{\sqrt{\alpha_{x|_{x=L}}}} |\psi_{x|_{x=L}}| \sqrt{\alpha_{x|_{x=L}}} |\psi_{xx|_{x=L}}| dt \\ &\leq 3 \frac{a^2}{(b-2a)} \frac{T^2}{L^4 s^2} \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt + \frac{3}{2} \varepsilon^2 s \int_0^T \alpha_{x|_{x=L}} |\psi_{xx|_{x=L}}|^2 dt \end{aligned}$$

Observe that dominating second term of $\mathcal{B}_L(\psi)$ will cancel out with the last term of the above inequality at the very of the proof. For the last term

$$\left| 2M\varepsilon s \int_0^T \alpha_{x|_{x=L}} |\psi_{x|_{x=L}}|^2 dt \right| \leq 2(b-2a) \frac{|M| T^2}{L^2 \varepsilon s^2} \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt.$$

Gathering the above inequalities we get

$$\mathcal{B}(\psi, L) \geq \mathcal{B}_1(s) \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt$$

where

$$\mathcal{B}_1(s) = 4(b-2a)^3 - \frac{1}{4}(b+c-a)\frac{T}{\varepsilon L s^2} - 3\frac{a^2}{(b-2a)}\frac{T^2}{L^4 s^2} - 2(b-2a)\frac{|M|T^2}{L^2 \varepsilon s^2}.$$

Since s satisfies (3.5), we get

$$\mathcal{B}_1(s) \geq 4(b-2a)^3 - \frac{1}{4}(b+c-a)\frac{1}{C_1^2} - 3\frac{a^2}{(b-2a)}\frac{1}{C_3^2} - 2(b-2a)\frac{1}{C_2^2}.$$

Choosing the polynomial. In what follows C will be a generic positive constant independent of ε, L, s and M. If we fix C_1, $C_2 \leq 1$ and C_3 large enough as needed, it is easy to construct some quadratic polynomial yielding positive constants $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{B}_1$, take for instance $\beta(x) = -x^2 + 4Lx + L^2$. Then we readily get

$$\mathcal{D}(\psi) \geq C \left(\varepsilon^2 \iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \varepsilon^2 \iint_Q L^2 s^3 \rho^3 |\psi_x|^2 dxdt \right)$$

and $\mathcal{B}_L(\psi) \geq 0$.

Residue term. From the previous discussion we deduce the following inequality

$$\begin{aligned} 2\iint_Q \mathcal{L}_1\psi \mathcal{L}_2\psi dxdt + \frac{3}{2}\varepsilon^2 s \int_0^T \alpha_{x|_{x=0}} |\psi_{xx|_{x=0}}|^2 dt \\ \geq C \left(\varepsilon^2 \iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \varepsilon^2 \iint_Q L^3 s^3 \rho^3 |\psi_x|^2 dxdt \right). \quad (3.6) \end{aligned}$$

Since $\alpha_{xxx} = 0$, for the residue term we have

$$\|\mathcal{R}\psi\|^2_{L^2(Q)} = 3\varepsilon^2 s^4 \iint_Q (\alpha_x)^2 (\alpha_{xx})^2 |\psi|^2 dxdt \leq C \frac{T}{L^2 s} \iint_Q \varepsilon^2 L^4 s^5 \rho^5 |\psi|^2 dxdt,$$

which can be absorbed by the left-hand side of (3.6) since $s \geq C_3 T L^{-2}$.

From (3.3) we obtain

$$C \left(\varepsilon^2 \iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \varepsilon^2 \iint_Q L^2 s^3 \rho^3 |\psi_x|^2 dxdt \right) \leq \|\mathcal{P}\psi\|^2_{L^2(Q)} + \varepsilon^2 s \int_0^T \alpha_{x|_{x=0}} |\psi_{xx|_{x=0}}|^2 dt, \quad (3.7)$$

for all s satisfying (3.5).

Conclusion. Recall that $\psi = e^{-s\alpha}\varphi$. First of all, since $\varphi_{|x=0} = \varphi_{x|x=L} = 0$ we obtain that

$$|\psi_{xx|x=0}|^2 = e^{-2s\alpha}|\varphi_{xx|x=0}|^2$$

which implies directly

$$\int_0^T \alpha_{x|x=0}|\psi_{xx|x=0}|^2 dt = \int_0^T \alpha_{x|x=0} e^{-2s\alpha|_{x=0}}|\varphi_{xx|x=0}|^2 dt.$$

On the other hand, we have the estimate

$$e^{-2s\alpha}|\varphi_x|^2 \leq C(s^2\rho^2|\psi|^2 + |\psi_x|^2).$$

Then (3.7) implies

$$C\left(\iint_Q L^4 s^5\rho^5 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q L^2 s^3\rho^3 e^{-2s\alpha}|\varphi_x|^2 dxdt\right) \leq \frac{1}{\varepsilon^2}\|\mathcal{P}\psi\|^2_{L^2(Q)} + \int_0^T Ls\rho e^{-2s\alpha|_{x=0}}|\varphi_{xx|x=0}|^2 dt. \quad (3.8)$$

Since $\mathcal{P}\psi = 0$, from (3.8) we get the desired inequality for φ.

3.2. **Carleman estimate with observation at $x = L$.** We consider the same structure of weight function (3.1) introduced previously, but here we suppose that β is a strictly positive, strictly decreasing and concave quadratic polynomial. With this modification we obtain the following Carleman estimate for the adjoint system (2.3) with observation at $x = L$.

Proposition 11. *There exists positive constants C, C_1, C_2 and C_3 independent of ε, L, s and $M \in \mathbb{R}$ such that for any $\varphi_T \in L^2(0, L)$ we have*

$$C\left(\iint_Q L^4 s^5\rho^5 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q L^2 s^3\rho^3 e^{-2s\alpha}|\varphi_x|^2 dxdt\right) \leq \int_0^T L^3 s^3\rho^3 e^{-2s\alpha|_{x=L}}|\varphi_{x|x=L}|^2 dt + \int_0^T Ls\rho e^{-2s\alpha|_{x=L}}|\varphi_{xx|x=L}|^2 dt, \quad (3.9)$$

for any $s \geq C_1 T^{1/2} L^{-1/2}\varepsilon^{-1/2} + C_2 TL^{-1}\varepsilon^{1/2}|M|^{1/2} + C_3 TL^{-2}$, where φ is the solution of (2.3) associated to φ_T.

Proof of Proposition 11. The proof is essentially the same as Proposition 9, therefore we point out the main differences.

Weight function. Let α be as in (3.1) with $\beta(x) := -ax^2 - bLx + cL^2$ an strictly positive, strictly decreasing and concave polynomial of degree 2. Then a, b, c are positive real numbers and α must satisfy the following inequalities in Q

$$L^2(c-a-b)\rho \leq \alpha \leq L^2 c\rho, \; -L(2a+b)\rho \leq \alpha_x \leq -Lb\rho, \; \alpha_{xx} = -2a\rho.$$

Also we impose that $c > a + b$. Concerning to the time derivatives, using $2 \leq T\rho$ we have the following estimates

$$|\alpha_t| \leq \frac{c}{2}TL^2\rho^3, \; |\alpha_{xt}| \leq \frac{(2a+b)}{2}TL\rho^3, \; |\alpha_{xxt}| \leq aT\rho^3, \; |\alpha_{tt}| \leq \frac{3}{4}cT^2L^2\rho^5.$$

Distributed terms. Similar computations as in the proof of Proposition 9 lead us to

$$\mathcal{D}(\psi) \geq \mathcal{D}_0(s)\varepsilon^2 \iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \mathcal{D}_1(s)\varepsilon^2 \iint_Q L^3 s^3 \rho^3 |\psi_x|^2 dxdt, \tag{3.10}$$

where

$$\mathcal{D}_0(s) = 14ab^4 - \frac{3}{2}(2a+b)^3 \frac{T}{L\varepsilon s^2} - 3ac(2a+b)\frac{T}{L\varepsilon s^2} - \frac{3}{2}a(2a+b)^2 \frac{|M|T^2}{L^2\varepsilon s^2} - \frac{3}{4}c\frac{T^2}{L^2\varepsilon^2 s^4} - \frac{(2a+b)}{8}\frac{T^3|M|}{L^3\varepsilon^2 s^4},$$
$$\mathcal{D}_1(s) = 6ab^2 - \frac{3}{2}(2a+b)\frac{T}{L\varepsilon s^2} - \frac{3}{4}a\frac{T^2|M|}{L^2\varepsilon s^2}.$$

Let s fulfilling

$$s \geq C_1 T^{1/2} L^{-1/2} \varepsilon^{-1/2} + C_2 T L^{-1} \varepsilon^{1/2} |M|^{1/2} + C_3 T L^{-2}, \tag{3.11}$$

where C_1, $C_2 > 0$ and $C_3 > 0$ will be chosen later. This choice of s implies that

$$\mathcal{D}_0(s) \geq 14ab^4 - \frac{3}{2}(2a+b)^3 \frac{1}{C_1^2} - 3ac(2a+b)\frac{1}{C_1^2} - \frac{3}{2}a(2a+b)^2 \frac{1}{C_2^2} - \frac{3}{4}c\frac{1}{C_1^4} - \frac{(2a+b)}{8}\frac{1}{C_1^2 C_2^2},$$
$$\mathcal{D}_1(s) \geq 6ab^2 - \frac{3}{2}(2a+b)\frac{1}{C_1^2} - \frac{3}{4}a\frac{1}{C_2^2}.$$

Boundary terms. Observe that $\mathcal{B}_0(x)$ is a positive since $-\alpha_x > 0$. In the same way, the dominating first order term at $x = L$ is negative

$$0 \geq 4\varepsilon^2 s^3 \int_0^T \alpha^3_{x|_{x=L}} |\psi_{x|_{x=L}}|^2 dxdt \geq -C \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dxdt.$$

Straightforward estimates implies

$$|\mathcal{B}_L(\psi)| \leq C\left(\int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt + \int_0^T \varepsilon^2 L s\rho |\psi_{xx|_{x=L}}|^2 dt\right).$$

Residue term. Let $C_3 > 0$ be large enough as needed. Then from the previous estimates we deduce the following inequality

$$2\iint_Q \mathcal{L}_1\psi\mathcal{L}_2\psi dxdt + \int_0^T \varepsilon^2 L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt + \int_0^T \varepsilon^2 L s\rho |\psi_{xx|_{x=L}}|^2 dt$$
$$\geq C\left(\varepsilon^2 \iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \varepsilon^2 \iint_Q L^3 s^3 \rho^3 |\psi_x|^2 dxdt\right). \tag{3.12}$$

Now, since $\alpha_{xxx} = 0$ we have that

$$\|\mathcal{R}\psi\|^2_{L^2(Q)} = 3\varepsilon^2 s^4 \iint_Q (\alpha_x)^2 (\alpha_{xx})^2 |\psi|^2 dxdt \leq C\frac{T}{L^2 s} \iint_Q \varepsilon^2 L^4 s^5 \rho^5 |\psi|^2 dxdt,$$

which can be absorbed by the left-hand side of (3.12) since $s \geq C_3 T L^{-2}$. From (3.3) we obtain

$$C\left(\iint_Q L^4 s^5 \rho^5 |\psi|^2 dxdt + \iint_Q L^2 s^3 \rho^3 |\psi_x|^2 dxdt\right)$$
$$\leq \frac{1}{\varepsilon^2}\|\mathcal{P}\psi\|^2_{L^2(Q)} + \int_0^T L^3 s^3 \rho^3 |\psi_{x|_{x=L}}|^2 dt + \int_0^T L s\rho |\psi_{xx|_{x=L}}|^2 dt \tag{3.13}$$

for all s satisfying (3.11).

Conclusion. The conclusion follows as in the proof of Proposition (9), by using that $\psi = e^{-s\alpha}\varphi$ together with the boundary conditions of φ and the properties of the weight function α.

4. Uniform cost of the null controllability

This section is devoted to prove Theorems 1 and 3, mainly following the ideas used in [10, Theorem 1.1]. To that end, we will first obtain some inequalities for the solutions of the adjoint system (2.3).

4.1. **Exponential dissipation.** Here we obtain an exponential dissipation inequality for the solutions of (2.3) by replicating the proof given in [10, Proposition 3.2]. Although the dissipation inequality obtained in [10] is still valid for equation (2.3), the novelty is that, here we slightly improve the constant appearing on the exponential upper bound of the dissipation with respect to the one established there and coincides with the one obtained in [8, Proposition 6], which is based on the fundamental solution of the equation $v_t + \varepsilon v_{xxx} = 0$. Furthermore, it is consistent with both the results from [10] and [8] in the amount of time required to have an exponential dissipation. In that sense, the following estimate seems optimal.

Proposition 12. *Let $T > 0$, $\varepsilon > 0$ and $M > 0$. Let $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 \ge L/M$. We have the following decay property:*

$$\int_0^L |\varphi(t_1,x)|^2 dx \le \exp\left\{-\frac{2}{3\sqrt{3}} \frac{(M(t_2-t_1)-L)^{3/2}}{\varepsilon^{1/2}(t_2-t_1)^{1/2}}\right\} \int_0^L |\varphi(t_2,x)|^2 dx, \tag{4.1}$$

for any solution φ of the adjoint system (2.3)

Proof. Let us consider the multiplier $\exp\{r(M(T-t)-x))\}\varphi$, where $r > 0$ will be chosen below. Then, integrating in $(0,L)$ and integrating by parts with respect to x, we have

$$-\varepsilon \int_0^L \exp\{r(M(T-t)-x))\}\varphi\varphi_{xxx} dx = -\frac{\varepsilon r^3}{2}\int_0^L \exp\{r(M(T-t)-x))\}|\varphi|^2 dx + \frac{3\varepsilon r}{2}\int_0^L \exp\{r(M(T-t)-x))\}|\varphi_x|^2 dx + \frac{\varepsilon}{2}\exp\{r(M(T-t)-L))\}|\varphi_{x|_{x=L}}|^2,$$

and

$$M\int_0^L \exp\{r(M(T-t)-x))\}\varphi\varphi_x dx = \frac{Mr}{2}\int_0^L \exp\{r(M(T-t)-x))\}|\varphi|^2 dx.$$

From the last identity we deduce

$$\int_0^L \exp\{r(M(T-t)-x))\}\varphi(-\varphi_t + M\varphi_x) dx = -\frac{1}{2}\frac{d}{dt}\int_0^L \exp\{r(M(T-t)-x))\}|\varphi|^2 dx.$$

Putting together the above computations and using that φ satisfies (2.3) we get

$$-\frac{1}{2}e^{-\varepsilon r^3(T-t)}\left(\frac{d}{dt}\int_0^L \exp\{r(M(T-t)-x))\}|\varphi|^2 dx + \varepsilon r^3 \int_0^L \exp\{r(M(T-t)-x))\}|\varphi|^2 dx\right)$$
$$= -\frac{\varepsilon}{2}e^{-\varepsilon r^3(T-t)}\left(3r\int_0^L \exp\{r(M(T-t)-x))\}|\varphi_x|^2 dx + \exp\{r(M(T-t)-L))\}|\varphi_{x|_{x=L}}|^2\right).$$

The previous identity allows us to deduce the differential inequality

$$-\frac{d}{dt}\left(\exp\{-\varepsilon r^3(T-t)\}\int_0^L e^{r(M(T-t)-x))}|\varphi|^2 dx\right) \le 0,$$

for $t \in (0,T)$. By considering $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 \ge L/M$, integrating between t_1 and t_2 the above differential inequality

$$\int_0^L |\varphi(t_1,x)|^2 dx \le \exp\{\varepsilon(t_2-t_1)r^3 - (M(t_2-t_1)-L)r\}\int_0^L |\varphi(t_2,x)|^2 dx.$$

Denoting by $K = K(r)$ the exponential factor appearing in the above inequality, we observe that the quantity

$$r^* = \left(\frac{M(t_2-t_1)-L}{3\varepsilon(t_2-t_1)}\right)^{1/2}$$

minimizes $K(r)$. Therefore K achieves its minimum at

$$K^* = \exp\left\{-\frac{2}{3\sqrt{3}}\frac{(M(t_2-t_1)-L)^{3/2}}{\varepsilon^{1/2}(t_2-t_1)^{1/2}}\right\}$$

and the conclusion follows. □

4.2. **Proof of Theorem 1.** The goal here is to prove an observability inequality like (2.4). Let $\widetilde{Q} := [\eta T, \kappa T] \times [0, L]$, where $1/2 < \eta < \kappa < 1$ will be chosen later. Also we consider s fulfilling the assumptions of the Carleman estimate, namely,

$$s = C_1 T^{1/2} L^{-1/2} \varepsilon^{-1/2} + C_2 T L^{-1} \varepsilon^{1/2} M^{1/2} + C_3 T L^{-2}, \tag{4.2}$$

where C_1, C_2 and C_3 will be chosen later on. In what follows, C will represent some positive constant independent of ε, L, s and M.

4.2.1. *Observability inequality.* Let $\beta(x) = -ax^2 + bLx + cL^2$ be as in (3.1) with a, b, $c > 0$. We have that

$$\alpha(t, x) \leq \mathfrak{g}(\kappa)(b + c - a)\frac{L^2}{T} \tag{4.3}$$

for all $(t, x) \in \widetilde{Q}$, where $\mathfrak{g}(\kappa) := (\kappa(1-\kappa))^{-1/2}$. On the other hand, the inequality $e^x \geq x^m/m!$, which holds for all $x \geq 0$ and $m \in \mathbb{N}$, along with the fact that $\alpha(t, 0) = cL^2\rho(t)$ implies

$$e^{2s\alpha(t,0)} \geq \frac{1}{m!}\left(\frac{s}{2}cL^2\rho(t)\right)^m e^{\frac{3}{2}cL^2 s\rho(t)}.$$

The choice of $m = 1$ in the above inequality combined with the fact that $\rho \geq 2/T$ and (4.3), allows us to deduce from the Carleman estimate (3.2) that

$$2^5 L^4 \frac{s^5}{T^5} e^{-2\mathfrak{g}(\kappa)(b+c-a)sL^2/T} \iint_{\widetilde{Q}} |\varphi|^2 dxdt \leq \frac{C}{L} e^{-3csL^2/T} \int_0^T |\varphi_{xx|_{x=0}}|^2 dt.$$

By the relation

$$\frac{s}{T} = \frac{1}{\varepsilon^{1/2} T^{1/2}}\left(C_1 L^{-1/2} + C_2 \frac{T^{1/2} M^{1/2}}{L}\right) + C_3 L^{-2}$$

we readily get inequality

$$\iint_{\widetilde{Q}} |\varphi|^2 dxdt \leq CL^5 \exp\left\{\left(2\mathfrak{g}(\kappa)(b+c-a) - 3c\right)\left(C_1 L^{3/2} + C_2 T^{1/2} L M^{1/2}\right)\right\} \int_0^T |\varphi_{xx|_{x=0}}|^2 dt. \tag{4.4}$$

Now we suppose that $T > 0$ satisfies $\eta T \geq L/M$ for some $\eta > 0$ to be chosen and let $t \in [\eta T, \kappa T]$. By taking $t_1 = 0$ and $t_2 = t$ in Proposition 12 we get

$$\begin{aligned}\int_0^L |\varphi(0, x)|^2 dx &\leq \exp\left\{-\frac{2}{3\sqrt{3}} \frac{(Mt - L)^{3/2}}{\varepsilon^{1/2} t^{1/2}}\right\} \int_0^L |\varphi(t, x)|^2 dx \\ &\leq \exp\left\{-\frac{2}{3\sqrt{3}} \frac{(\eta MT - L)^{3/2}}{\varepsilon^{1/2} \kappa^{1/2} T^{1/2}}\right\} \int_0^L |\varphi(t, x)|^2 dx. \end{aligned} \tag{4.5}$$

Integrating (4.5) in $[\eta T, \kappa T]$ and using (4.4) we obtain the observability inequality

$$\int_0^L |\varphi_{|t=0}|^2 dx \leq C\frac{L^5}{T} \exp\left\{\frac{\mathcal{K}(T)}{\varepsilon^{1/2} T^{1/2}}\right\} \int_0^T |\varphi_{xx|_{x=0}}|^2 dt \tag{4.6}$$

where

$$\mathcal{K}(T) := \left(2\mathfrak{g}(\kappa)(b+c-a) - 3c\right)\left(C_1 L^{3/2} + C_2 T^{1/2} L M^{1/2}\right) - \frac{2}{3\sqrt{3}\kappa^{1/2}}(\eta MT - L)^{3/2}.$$

4.2.2. *Explicit time.* Now let $T = \tau L/M$ where $\tau > 1/\eta$. Then

$$\mathcal{K}(\tau L/M) = L^{3/2}\left[\left(2\mathfrak{g}(\kappa)(b+c-a) - 3c\right)\left(C_1 + C_2\tau^{1/2}\right) - \frac{2}{3\sqrt{3}\kappa^{1/2}}(\tau\eta - 1)^{3/2}\right]$$

The idea now is to choose the parameters in such a way that the negative part of $\mathcal{K}$ counteracts its positive part to obtain $\mathcal{K} < 0$. However, this choice has to ensure that the Carleman estimate (3.2) holds. Then, in order to choose the polynomial β together with the parameters C_1, C_2 and C_3, we introduce the following constraint functions

$$g_1(a, b, c, C_1, C_2) = 14a(b - 2a)^4 - \frac{3}{2}b^3\frac{1}{C_1^2} - \frac{3}{2}(b + c - a)ab\frac{1}{C_1^2} - \frac{3}{2}ab^2\frac{1}{C_2^2} - \frac{3}{4}(b + c - a)\frac{1}{C_1^4} - \frac{b}{8}\frac{1}{C_1^2 C_2^2},$$

$$g_2(a,b,c,C_1,C_2) = 6a(b-2a)^2 - \frac{3}{2}b\frac{1}{C_1^2} - \frac{3}{4}a\frac{1}{C_2^2},$$
$$g_3(a,b,c,C_1,C_2) = 4(b-2a)^3 - \frac{1}{4}(b+c-a)\frac{1}{C_1^2} - 3\frac{a^2}{(b-2a)}\frac{1}{C_3^2} - 2(b-2a)\frac{1}{C_2^2},$$
$$g_4(a,b,c,C_1,C_2) = b-2a,$$

and then we consider the following nonlinear optimization problem

$$\min_{(a,b,c,C_1,C_2)\in \boldsymbol{X}} (4b+c-4a)^2(C_1^2+C_2^2) \tag{4.7}$$

where

$$\boldsymbol{X} = \{(a,b,c,C_1,C_2) \in \mathbb{R}_+^5 : g_1 > 0,\ g_2 > 0,\ g_3 > 0,\ g_4 > 0 \text{ and } C_1, C_2 \le 1\}.$$

The minimization problem (4.7) is a proposal to reduce the effect of the positive part, where the objective function minimizes simultaneously the factors accompanying τ^0 and $\tau^{1/2}$. Observe that we replace $\kappa \mapsto \mathfrak{g}(\kappa)$ by its lower bound 2 to obtain the objective function, which in such case only depends on the parameters coming from the Carleman estimate. The constraints in $\boldsymbol{X}$ ensures that the obtained polynomial is eligible as a weight function. On the other hand, to increase the effect of the negative part of $\mathcal{K}$, we will choose η and κ, as large and small as possible, respectively. Since $\eta < \kappa$, the difference between them must be small. It is worth mentioning that any other proposal of such η and κ does not improve the result obtained for τ.

Now, if we fix $C_3 = 10^{7/2}$, we obtain the following approximated solution of (4.7)

$$a = 0.4768,\ b = 4.2744,\ c = 0.01,\ C_1 = 1/2,\ C_2 = 1/4.0622.$$

By inspection one can see that by choosing $\eta = 0.76$ and $\kappa = 0.7601$, the map $\tau \mapsto \mathcal{K}(\tau L/M)$ is decreasing in $[23.25, +\infty)$ and satisfies

$$\mathcal{K}(23.25L/M) \le -CL^{3/2}.$$

By the above observation we have that $\mathcal{K}(\tau L/M) \le \mathcal{K}(23.25L/M)$ for $\tau \ge 23.25$. Using the standard HUM method, from (4.6) we arrive the desired inequality (1.2).

4.3. **Proof of Theorem 3.** The proof is essentially the same as the proof of Theorem 1, that is, we will prove an observability inequality like (2.5). Let $\widetilde{Q} := [\eta T, \kappa T] \times [0, L]$, where $1/2 < \eta < \kappa < 1$ will be chosen later and consider s fulfilling the assumptions of the Carleman estimate.

4.3.1. *Observability inequality.* As was pointed out previously, let $\beta(x) = -ax^2 - bLx + cL^2$ be an strictly positive, strictly decreasing concave polynomial with a, b, $c > 0$. Then we have

$$\alpha(t,x) \le \mathfrak{g}(\kappa) c \frac{L^2}{T} \tag{4.8}$$

for all $(t,x) \in \widetilde{Q}$, where $\mathfrak{g}(\kappa) = (\kappa(1-\kappa))^{-1/2}$. On the other hand, for all $m \in \mathbb{N}$ the following inequality holds

$$e^{2s\alpha(t,L)} \ge \frac{1}{m!}\left(\frac{s}{2}(c-a-b)L^2\rho(t)\right)^m e^{3(c-a-b)L^2 s/T},$$

for all $t \in (0,T)$. By choosing $m = 1$ and $m = 3$ in the above inequality, when applied with (4.8) to the Carleman estimate (3.9) we get

$$\iint_{\widetilde{Q}} |\varphi|^2 dxdt \le C(L^3+L^5)\exp\left\{\frac{\left(2\mathfrak{g}(\kappa)c - 3(c-a-b)\right)}{\varepsilon^{1/2}T^{1/2}}\left(C_1 L^{3/2} + C_2 T^{1/2} L M^{1/2}\right)\right\}$$
$$\left(\int_0^T |\varphi_{x|_{x=L}}|^2 dt + \int_0^T |\varphi_{xx|_{x=L}}|^2 dt\right). \tag{4.9}$$

Now we suppose that $T > 0$ satisfies $\eta T \ge L/M$ for some $\eta > 0$. Integrating the exponential dissipation estimate (4.5) in $[\eta T, \kappa T]$ and using (4.9) we obtain the observability inequality

$$\int_0^L |\varphi_{|_{t=0}}|^2 dx \le C\left(\frac{L^3+L^5}{T}\right)\exp\left\{\frac{\mathcal{K}(T)}{\varepsilon^{1/2}T^{1/2}}\right\}\left(\int_0^T |\varphi_{x|_{x=L}}|^2 dt + \int_0^T |\varphi_{xx|_{x=L}}|^2 dt\right) \tag{4.10}$$

where

$$\mathcal{K}(T) = \big(2\mathfrak{g}(\kappa)c - 3(c-a-b)\big)\left(C_1 L^{3/2} + C_2 T^{1/2} L M^{1/2}\right) - \frac{2}{3\sqrt{3}\kappa^{1/2}}(\eta M T - L)^{3/2}.$$

4.3.2. *Explicit time.* Here, we follow the same approach as in Section 4.2.2 adapted to Carleman estimate (3.9). Let $T = \tau L/M$ where $\tau > 1/\eta$. Then

$$\mathcal{K}(\tau L/M) = L^{3/2}\left[\big(2\mathfrak{g}(\kappa)c - 3(c-a-b)\big)\left(C_1 + C_2\tau^{1/2}\right) - \frac{2}{3\sqrt{3}\kappa^{1/2}}(\tau\eta - 1)^{3/2}\right].$$

Let define the constraints

$$g_1(a,b,c,C_1,C_2) = 14ab^4 - \frac{3}{2}(2a+b)^3\frac{1}{C_1^2} - 3ac(2a+b)\frac{1}{C_1^2} - \frac{3}{2}a(2a+b)^2\frac{1}{C_2^2} - \frac{3}{4}c\frac{1}{C_1^4} - \frac{(2a+b)}{8}\frac{1}{C_1^2C_2^2},$$

$$g_2(a,b,c,C_1,C_2) = 6ab^2 - \frac{3}{2}(2a+b)\frac{1}{C_1^2} - \frac{3}{4}a\frac{1}{C_2^2},$$

$$g_3(a,b,c,C_1,C_2) = c - a - b,$$

and then consider the following nonlinear optimization problem

$$\min_{(a,b,c,C_1,C_2)\in \boldsymbol{X}} (3a+3b+c)^2(C_1^2+C_2^2) \tag{4.11}$$

where

$$\boldsymbol{X} = \{(a,b,c,C_1,C_2) \in \mathbb{R}^5_+ : g_1 > 0,\ g_2 > 0,\ g_3 > 0 \text{ and } C_1, C_2 \le 1\}.$$

An approximated solution for this problem is

$$a = 0.7283,\ b = 5.463,\ c = 6.1919,\ C_1 = 1/3.2538,\ C_2 = 1/6.1279.$$

By inspection one can see that choosing $\eta = 0.76$ and $\kappa = 0.7601$, the map $\tau \mapsto \mathcal{K}(\tau L/M)$ is decreasing in $[24.31, +\infty)$ and satisfies

$$\mathcal{K}(24.31L/M) \le -CL^{3/2}.$$

By the above observation we have that $\mathcal{K}(\tau L/M) \le \mathcal{K}(24.31L/M)$ for $\tau \ge 24.31$. Using the classical duality result, from (4.10) we arrive the desired inequality (1.3) and the conclusion follows.

Remark 13. It is not directly clear whether or not the proposed optimization problems (4.7) and (4.11) admit a solution due to the nature of the constraints. In this regard, the existence of solutions of both problems was obtained numerically.

4.4. **Some comments about the method.** The method employed to obtain the minimal parameter τ, where $T = \tau L/M$, is not optimal in the sense that the Carleman estimates does not provide sharp estimates on the constants, so we cannot expect to obtain an optimal result nor make a conjecture of which would be such minimal τ. From [10, Theorem 1.4], the cost of null controllability explodes exponentially if $\tau < 1$. However, it is an open problem to know what is the behavior of the cost of null controllability when $\tau \in [1, 23.25)$ with u_1 being the active control, and when $\tau \in [1, 24.31)$ with controls u_2 and u_3.

In view of the results obtained by Glass [9] for the vanishing diffusion heat equation $y_t - \varepsilon y_{xx} + My_x = 0$, the moments method is a good approach to the problem. However, the steady-state operator associated to equation (1.1) does not admit a well-behaved spectral decomposition. Up to our knowledge, the last improvement for the minimal time of uniform null controllability for this case, was obtained by Lissy [13], with a proof based on the link between the cost of the controls of the heat equation $y_t - y_{xx} = 0$ and the cost of the null controllability for the vanishing diffusion heat equation. In the same way, Lissy [15] also obtained the best known upper bound for the explosion time of the cost of null controllability.

In the case of dispersive equations, Lissy [14] based on the moment method, studied the cost of fast controls of a family of dispersive equations, obtaining as a consequence results for a Korteweg-de Vries equation with periodic boundary conditions. For the equation (1.1) with $u_1 = u_2 = 0$ and $\varepsilon = 1, M = -1$ under the assumption that $T \ge T_0(L) > 0$ and a right Neumann control, Krieger and Xiang [11] studied the cost of controllability, but not providing information about the behavior of fast controls, namely, when $T \to 0$. The possibility of linking the cost of fast controls of some Korteweg-de Vries equation with the cost of the null controllability of equation (1.1) is an interesting problem.

5. Application to a fourth-order parabolic equation

Let T, $L > 0$. Here we consider the following linear equation

$$\begin{cases} y_t + \varepsilon y_{xxxx} + M y_x = 0, & (t,x) \in Q, \\ y(t,0) = u_1(t),\ y(t,L) = 0, & t \in (0,T), \\ y_{xx}(t,0) = u_2(t),\ y_{xx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ and $M \in \mathbb{R} \setminus \{0\}$ are the diffusion and the transport coefficients, u_1, u_2 are the controls and y_0 is the initial condition.

The control system (5.1) was studied by Carreño and Guzmán [3], where by using an exponential dissipation inequality together with a suitable Carleman estimate, the uniform null controllability is established. More precisely, the cost of null controllability goes to 0 when $\varepsilon \to 0^+$, provided that $T \geq 40L/M$ with $M \neq 0$. Adapting the approach of the present article we can improve the time from which the uniform null controllability holds, and then reformulate [3, Theorem 1.2].

For this purpose, we consider the weight function

$$\alpha(t,x) = \frac{-0.469x^2 + 6.592Lx + 0.01L^2}{t^{1/3}(T-t)^{1/3}},\ (t,x) \in Q.$$

We also consider $C_1 = 1/3.5$, $C_2 = 1/6.941$, $C_3 = 10^{7/2}$ and the cut interval $\widetilde{Q} = [0.807T, 0.8071T] \times (0,L)$. Let $V := H^2 \cap H_0^1(0,L)$ and denote by V^* its dual space, by identifying $L^2(0,L)$ with itself.

Theorem 14. *Let $T \geq 32.66L/|M|$ with $M \neq 0$. For every $y_0 \in L^2(0,L)$, there exists $(u_1,u_2) \in L^2(0,T)^2$ such that the unique solution (defined by transposition) $y \in C([0,T];V^*)$ of (5.1) satisfies $y(T,\cdot) = 0$ in V^*. Moreover, there exists C, $c > 0$ both independent of ε, T, L and M such that*

$$\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} \leq \frac{C}{\varepsilon^2}\left(\frac{L^3 + L^7}{T}\right)\exp\left\{-c\frac{L^{4/3}}{\varepsilon^{1/3}T^{1/3}}\right\}\|y_0\|^2_{L^2(0,L)}.$$

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